Calculating a Determinant with Condensation

In 1865, Charles Dodgson, better known for his pen name Lewis Carroll, published *Alice in Wonderful*. Two years later, this mathematician at Christ Church published a document on an algorithm for calculating determinants. The method, known as condensation, is useful for hand calculation of determinants. In this handout, we'll look at two examples, which allows us to walk through this process twice. First, we will consider a 3×3 matrix, which will be the size that appears on the quiz. Then, we'll look at a 4×4 matrix, which is the size we considered in the lecture.

Example 1

Let's walk through the steps to find the determinant of this matrix

$$A = \begin{bmatrix} 4 & 8 & 7 \\ 4 & 3 & 4 \\ 3 & 1 & 2 \end{bmatrix}$$

The 2×2 connected submatrices and their determinants from the first row of A are:

$$\begin{vmatrix} 4 & 8 \\ 4 & 3 \end{vmatrix} = 12 - 32 = -20 \text{ and } \begin{vmatrix} 8 & 7 \\ 3 & 4 \end{vmatrix} = 32 - 21 = 11.$$

The 2×2 connected submatrices and their determinants from the first row of A are:

$$\begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} = 4 - 9 = -5 \text{ and } \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2.$$

We use these determinants to form our 2×2 matrix, which is

$$B = \begin{bmatrix} -20 & 11 \\ -5 & 2 \end{bmatrix}.$$

Since this a 2×2 matrix, it is the only connected submatrix so we find its determinant

$$\begin{vmatrix} -20 & 11 \\ -5 & 2 \end{vmatrix} = -40 + 55 = 15.$$

We then divide this value by the central submatrix of the 3×3 matrix which equals 3. So the determinant of the matrix A is $15/3 = 5$.

Example 2

Now, let's walk through the steps to find the determinant of a 4×4 matrix

$$A = \begin{bmatrix} 5 & 3 & 2 & 4 \\ 2 & 5 & 1 & 1 \\ 7 & 4 & 2 & 8 \\ 5 & 1 & 2 & 8 \end{bmatrix}.$$

The 2×2 connected submatrices and their determinants from the first row of A are:

$$\begin{vmatrix} 5 & 3 \\ 2 & 5 \end{vmatrix} = 25 - 6 = 19, \begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = 3 - 10 = -7, \text{ and } \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} = 2 - 4 = -2.$$

The 2×2 connected submatrices and their determinants from the second row of A are:

$$\begin{vmatrix} 2 & 5 \\ 7 & 4 \end{vmatrix} = 8 - 35 = -27, \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = 10 - 4 = 6, \text{ and } \begin{vmatrix} 1 & 1 \\ 2 & 8 \end{vmatrix} = 8 - 2 = 6.$$

The 2×2 connected submatrices and their determinants from the third row of A are:

$$\begin{vmatrix} 7 & 4 \\ 5 & 1 \end{vmatrix} = 7 - 20 = -13, \begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} = 8 - 2 = 6, \text{ and } \begin{vmatrix} 2 & 8 \\ 2 & 8 \end{vmatrix} = 16 - 16 = 0.$$

We use these determinants to form our 3×3 matrix, which is

$$B = \begin{bmatrix} 19 & -7 & -2 \\ -27 & 6 & 6 \\ -13 & 6 & 0 \end{bmatrix}.$$

The 2×2 connected submatrices and their determinants from the first row of B are:

$$\begin{vmatrix} 19 & -7 \\ -27 & 6 \end{vmatrix} = 114 - 189 = -75 \text{ and } \begin{vmatrix} -7 & -2 \\ 6 & 6 \end{vmatrix} = -42 + 12 = -30.$$

The 2×2 connected submatrices and their determinants from the second row of B are:

$$\begin{vmatrix} -27 & 6 \\ -13 & 6 \end{vmatrix} = -162 + 78 = -84 \text{ and } \begin{vmatrix} 6 & 6 \\ 6 & 0 \end{vmatrix} = 0 - 36 = -36.$$

We use these determinants to form our 2×2 matrix, which is

$$C = \begin{bmatrix} -75 & -30 \\ -84 & -36 \end{bmatrix}.$$

We then divide each element by the corresponding element in the central submatrix of the 4×4 matrix A equaling

$$\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}.$$

That produces our 2×2 matrix

$$D = \begin{bmatrix} -75/5 & -24/1 \\ -84/4 & -36/2 \end{bmatrix} = \begin{bmatrix} -15 & -24 \\ -21 & -18 \end{bmatrix}.$$

Given D is a 2×2 matrix, we find it's determinant which equals -360. We then divide this value by the central submatrix of the 3×3 matrix B which equals 6. This equals $-360/6 = -60$.

So, the determinant of our 4×4 matrix A is -60.