Discrete Time Fourier Transform (DTFT)

Discrete Time Fourier Transform (DTFT)

- The DTFT is the Fourier transform of choice for analyzing infinite-length signals and systems
- Useful for conceptual, pencil-and-paper work, but not Matlab friendly (infinitely-long vectors)
- Properties are very similar to the Discrete Fourier Transform (DFT) with a few caveats
- We will derive the DTFT as the limit of the DFT as the signal length $N \to \infty$

Recall: DFT (Unnormalized)

- **Analysis (Forward DFT)**
  - Choose the DFT coefficients $X[k]$ such that the synthesis produces the signal x
  - The weight $X[k]$ measures the similarity between x and the harmonic sinusoid s_k
  - Therefore, $X[k]$ measures the "frequency content" of x at frequency k

$$X_u[k] \; = \; \sum_{n=0}^{N-1} x[n] \, e^{-j \frac{2\pi}{N} kn}$$

- **Synthesis (Inverse DFT)**
  - Build up the signal x as a linear combination of harmonic sinusoids s_k weighted by the DFT coefficients $X[k]$

$$x[n] \; = \; \frac{1}{N} \sum_{k=0}^{N-1} X_u[k] \, e^{j \frac{2\pi}{N} kn}$$

The Centered DFT

- Both $x[n]$ and $X[k]$ can be interpreted as periodic with period N, so we will shift the intervals of interest in time and frequency to be centered around $n, k = 0$

$$-\frac{N}{2} \;\leq\; n, k \;\leq\; \frac{N}{2} - 1$$

- The modified forward and inverse DFT formulas are

$$X_u[k] \;=\; \sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\,\frac{2\pi}{N}\,kn}, \qquad -\frac{N}{2} \;\leq\; k \;\leq\; \frac{N}{2} - 1$$

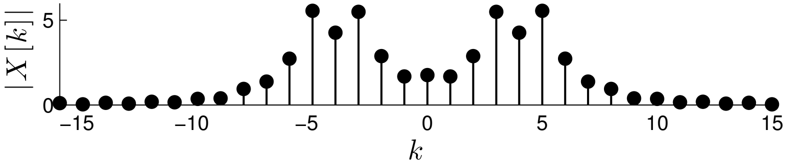
$$x[n] \;=\; \frac{1}{N} \sum_{k=-N/2}^{N/2-1} X_u[k]\, e^{j\,\frac{2\pi}{N}\,kn} \qquad -\frac{N}{2} \;\leq\; n \;\leq\; \frac{N}{2} - 1$$

Recall: DFT Frequencies

$$X_u[k] \;=\; \sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad -\frac{N}{2} \;\le\; k \;\le\; \frac{N}{2}-1$$

- $X_u[k]$ measures the similarity between the time signal x and the harmonic sinusoid s_k
- Therefore, $X_u[k]$ measures the "frequency content" of x at frequency

$$-\pi \;\le\; \omega_k = \frac{2\pi}{N}k \;<\; \pi$$

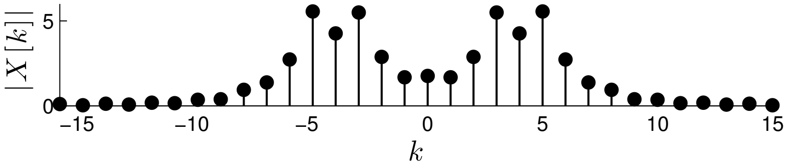


Take It To The Limit (1)

$$X_u[k] \;=\; \sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad -\frac{N}{2} \;\le\; k \;\le\; \frac{N}{2}-1$$

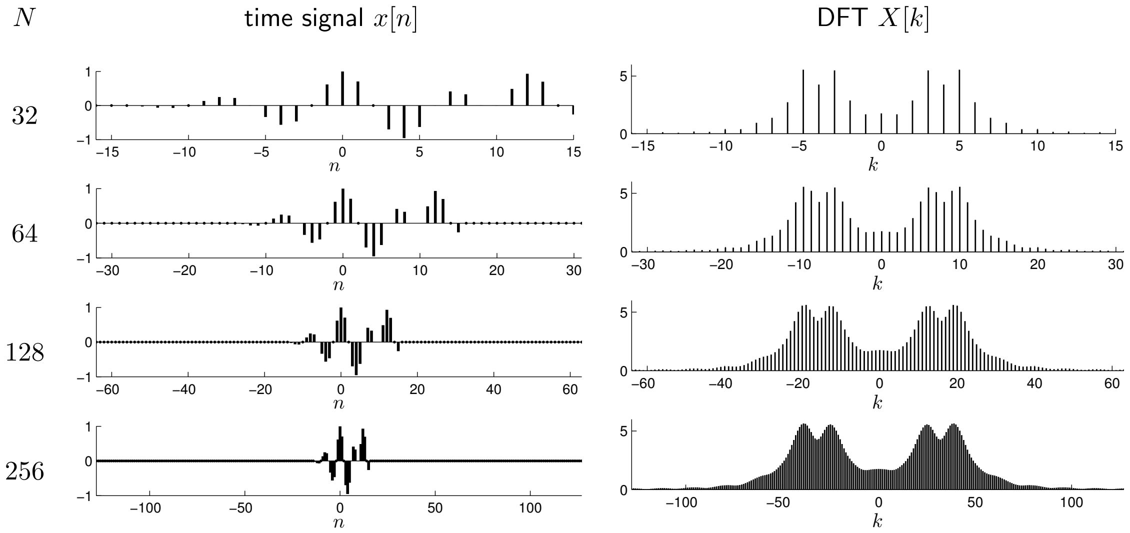
- Let the signal length N increase towards ∞ and study what happens to $X_u[k]$
- **Key fact:** No matter how large N grows, the frequencies of the DFT sinusoids remain in the interval

$$-\pi \;\le\; \omega_k = \frac{2\pi}{N}k \;<\; \pi$$



Take It To The Limit (2)

$$X_u[k] \;=\; \sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$



Discrete Time Fourier Transform (Forward)

- As $N \to \infty$, the forward DFT converges to a function of the **continuous frequency variable** ω that we will call the **forward discrete time Fourier transform** (DTFT)

$$\sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \longrightarrow \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \;=\; X(\omega), \qquad -\pi \leq \omega < \pi$$

- Recall: Inner product for infinite-length signals

$$\langle x, y \rangle \;=\; \sum_{n=-\infty}^{\infty} x[n]\, y[n]^*$$

- **Analysis interpretation:** The value of the DTFT $X(\omega)$ at frequency ω measures the similarity of the infinite-length signal $x[n]$ to the infinite-length sinusoid $e^{j\omega n}$

Discrete Time Fourier Transform (Inverse)

- Inverse unnormalized DFT

$$x[n] \;=\; \frac{2\pi}{2\pi N} \sum_{k=-N/2}^{N/2-1} X_u[k]\, e^{j\frac{2\pi}{N}kn}$$

- In the limit as the signal length $N \to \infty$, the inverse DFT converges in a more subtle way:

$$e^{j\frac{2\pi}{N}kn} \longrightarrow e^{j\omega n}, \qquad X_u[k] \longrightarrow X(\omega), \qquad \frac{2\pi}{N} \longrightarrow d\omega, \qquad \sum_{k=-N/2}^{N/2-1} \longrightarrow \int_{-\pi}^{\pi}$$

  resulting in the **inverse DTFT**

$$x[n] \;=\; \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}, \qquad \infty < n < \infty$$

- **Synthesis interpretation:** Build up the signal x as an infinite linear combination of sinusoids $e^{j\omega n}$ weighted by the DTFT $X(\omega)$

Summary

- Discrete-time Fourier transform (DTFT)

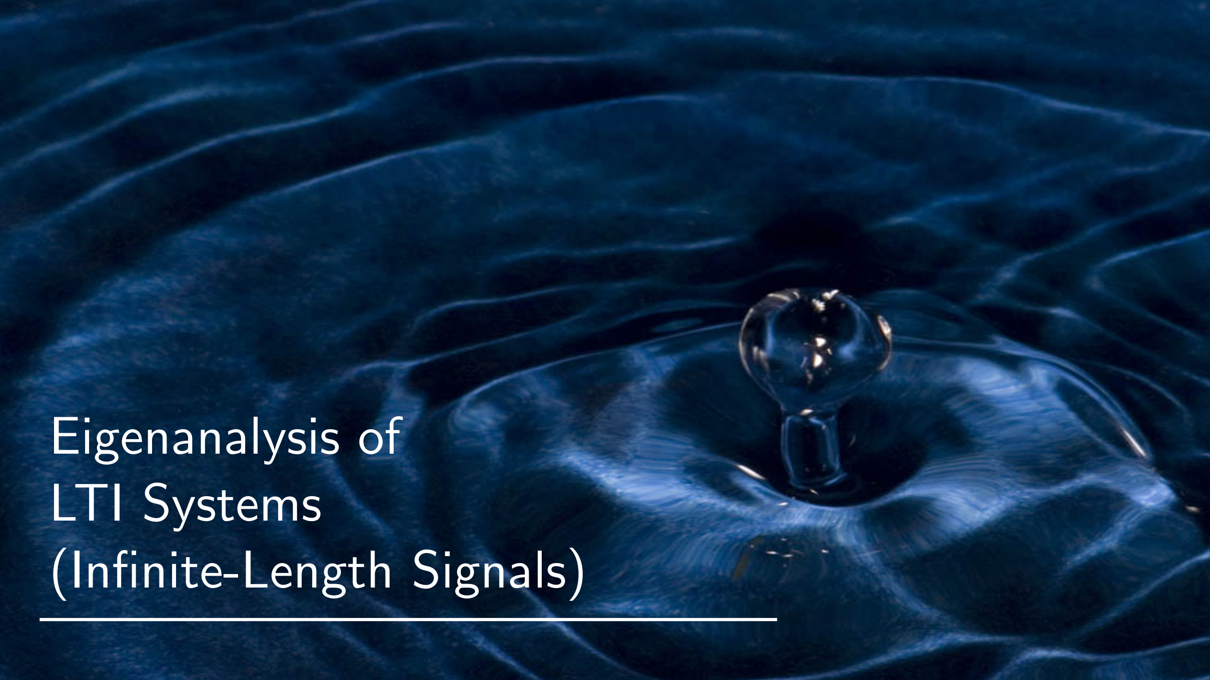
$$X(\omega) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad -\pi \leq \omega < \pi$$

$$x[n] \;=\; \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}, \qquad \infty < n < \infty$$

- The core "basis functions" of the DTFT are the sinusoids $e^{j\omega n}$ with arbitrary frequencies ω

- The DTFT can be derived as the limit of the DFT as the signal length $N \to \infty$

- The analysis/synthesis interpretation of the DFT holds for the DTFT, as do most of its properties



Eigenanalysis of LTI Systems (Infinite-Length Signals)

LTI Systems for Infinite-Length Signals

$$x \longrightarrow \boxed{\mathcal{H}} \longrightarrow y$$

$$y = \mathbf{H}x$$

- For infinite length signals, $\mathbf{H}$ is an infinitely large **Toeplitz matrix** with entries

$$[\mathbf{H}]_{n,m} = h[n-m]$$

where h is the **impulse response**

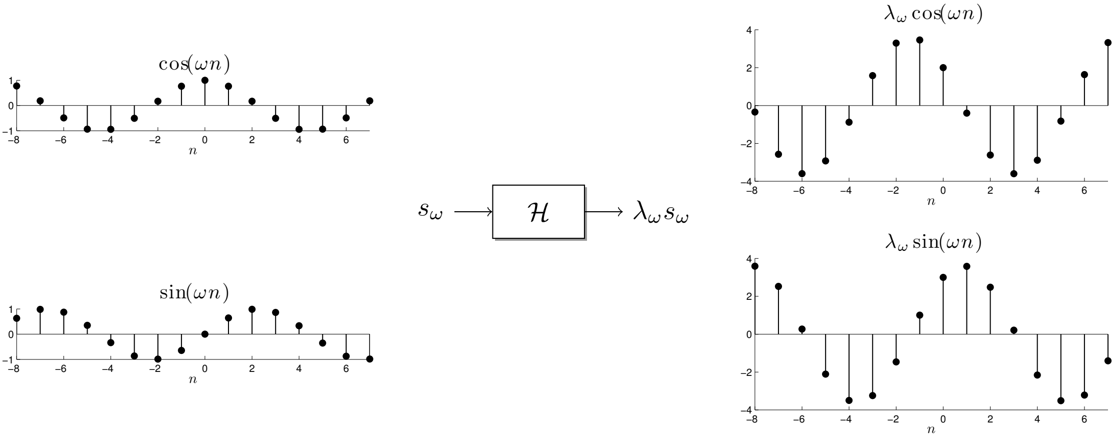
- **Goal:** Calculate the **eigenvectors** and **eigenvalues** of $\mathbf{H}$

- Eigenvectors v are input signals that emerge at the system output unchanged (except for a scaling by the eigenvalue λ) and so are somehow "fundamental" to the system

Eigenvectors of LTI Systems

- **Fact:** The eigenvectors of a Toeplitz matrix (LTI system) are the complex **sinusoids**

$$s_\omega[n] \;=\; e^{j\omega n} \;=\; \cos(\omega n) + j\sin(\omega n), \qquad -\pi \leq \omega < \pi, \quad -\infty < n < \infty$$



Sinusoids are Eigenvectors of LTI Systems

$$s_\omega \longrightarrow \boxed{\mathcal{H}} \longrightarrow \lambda_\omega s_\omega$$

- Prove that harmonic sinusoids are the eigenvectors of LTI systems simply by computing the convolution with input s_ω and applying the periodicity of the sinusoids (infinite-length)

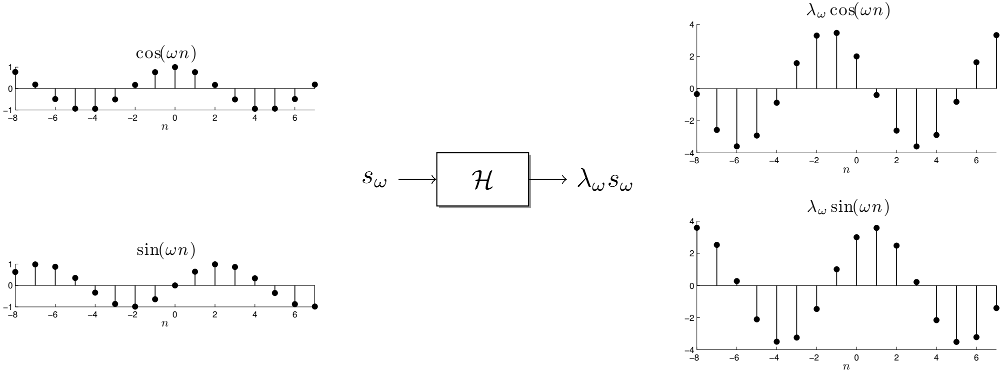
$$\begin{aligned}
s_\omega[n] * h[n] \quad &= \quad \sum_{m=-\infty}^{\infty} s_\omega[n-m]\, h[m] \; = \; \sum_{m=-\infty}^{\infty} e^{j\omega(n-m)}\, h[m] \\
&= \quad \sum_{m=-\infty}^{\infty} e^{j\omega n}\, e^{-j\omega m}\, h[m] \; = \; \left(\sum_{m=-\infty}^{\infty} h[m]\, e^{-j\omega m} \right) e^{j\omega n} \\
&= \quad \lambda_\omega\, s_\omega[n] \quad \checkmark
\end{aligned}$$

Eigenvalues of LTI Systems

- The eigenvalue $\lambda_\omega \in \mathbb{C}$ corresponding to the sinusoid eigenvector s_ω is called the **frequency response** at frequency ω since it measures how the system "responds" to s_k

$$\lambda_\omega \;=\; \sum_{n=-\infty}^{\infty} h[n]\, e^{-\omega n} \;=\; \langle h, s_\omega \rangle \;=\; H(\omega) \quad \text{(DTFT of } h\text{)}$$

- Recall properties of the **inner product**: λ_ω grows/shrinks as h and s_ω become more/less similar



Eigendecomposition and Diagonalization of an LTI System

$$x \longrightarrow \boxed{\mathcal{H}} \longrightarrow y$$

$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- While we can't explicitly display the infinitely large matrices involved, we can use the DTFT to "diagonalize" an LTI system

- Taking the DTFTs of x and h

$$X(\omega) = \sum_{m=-\infty}^{\infty} x[n]\, e^{-\omega n}, \qquad H(\omega) = \sum_{m=-\infty}^{\infty} h[n]\, e^{-\omega n}$$

  we have that

$$Y(\omega) \;=\; X(\omega) H(\omega)$$

  and then

$$y[n] = \int_{-\pi}^{\pi} Y(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

Summary

- Complex sinusoids are the eigenfunctions of LTI systems for infinite-length signals (Toeplitz matrices)

- Therefore, the discrete time Fourier transform (DTFT) is the natural tool for studying LTI systems for infinite-length signals

- Frequency response $H(\omega)$ equals the DTFT of the impulse response $h[n]$

- Diagonalization by eigendecomposition implies

$$Y(\omega) \;=\; X(\omega)\, H(\omega)$$

Discrete Time Fourier Transform Examples

Discrete Time Fourier Transform

$$
\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad -\pi \leq \omega < \pi \\
x[n] &= \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}, \qquad \infty < n < \infty
\end{aligned}
$$

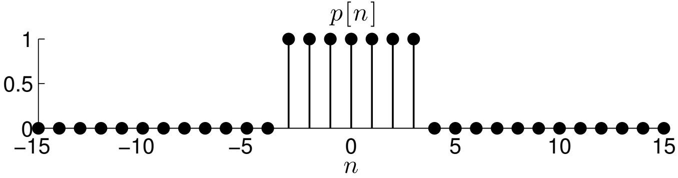
- The Fourier transform of choice for analyzing infinite-length signals and systems
- Useful for conceptual, pencil-and-paper work, but not Matlab friendly (infinitely-long vectors)

DTFT of the Unit Pulse (1)

- Compute the DTFT of the symmetrical **unit pulse** $p[n] = \begin{cases} 1 & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$

- Note: Duration $D_x = 2M + 1$ samples

- Example for $M = 3$



- Forward DTFT

$$P(\omega) \;=\; \sum_{n=-\infty}^{\infty} p[n]\, e^{-j\omega n} \;=\; \sum_{n=-M}^{M} e^{-j\omega n} \qquad \ldots$$

DTFT of the Unit Pulse (2)

- Apply the finite geometric series formula

$$P(\omega) = \sum_{n=-\infty}^{\infty} p[n]\, e^{-j\omega n} = \sum_{n=-M}^{M} e^{-j\omega n} = \sum_{n=-M}^{M} \left(e^{-j\omega}\right)^n = \frac{e^{j\omega M} - e^{-j\omega(M+1)}}{1 - e^{-j\omega}}$$

- This is an answer but it is not simplified enough to make sense, so we continue simplifying

$$\begin{aligned} P(\omega) &= \frac{e^{j\omega M} - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} = \frac{e^{-j\omega/2}\left(e^{j\omega\frac{2M+1}{2}} - e^{-j\omega\frac{2M+1}{2}}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} \\ &= \frac{2j\sin\left(\omega\frac{2M+1}{2}\right)}{2j\sin\left(\frac{\omega}{2}\right)} \end{aligned}$$

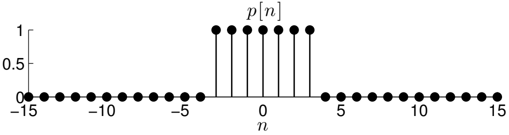
DTFT of the Unit Pulse (3)

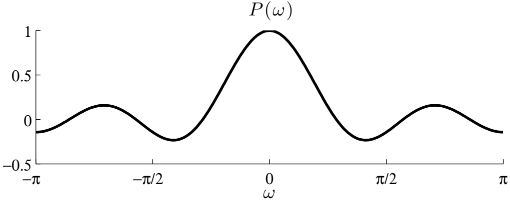
- Simplified DTFT of the unit pulse of duration $D_x = 2M + 1$ samples

$$P(\omega) \;=\; \frac{\sin\left(\frac{2M+1}{2}\,\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

- This is called the **Dirichlet kernel** or "digital sinc"
  - It has a shape reminiscent of the classical $\sin x/x$ sinc function, but it is 2π-periodic

- If $p[n]$ is interpreted as the impulse response of the moving average system, then $P(\omega)$ is the frequency response (eigenvalues) (low-pass filter)



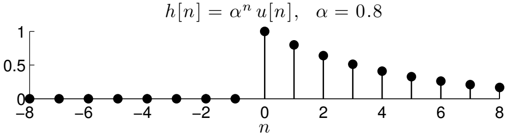


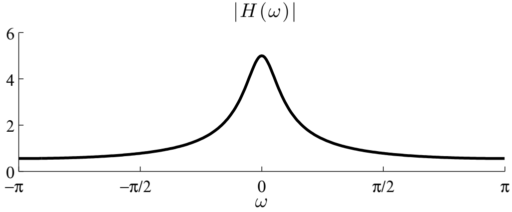
DTFT of a One-Sided Exponential

- Recall the impulse response of the recursive average system: $h[n] = \alpha^n \, u[n], \; |\alpha| < 1$
- Compute the frequency response $H(\omega)$
- Forward DTFT

$$H(\omega) \;=\; \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} \;=\; \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n} \;=\; \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n \;=\; \frac{1}{1-\alpha\, e^{-j\omega}}$$

- Recursive system with $\alpha = 0.8$ is a low-pass filter





Impulse Response of the Ideal Lowpass Filter (1)

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

- Compute the impulse response $h[n]$ given this $H(\omega)$

- Apply the inverse DTFT

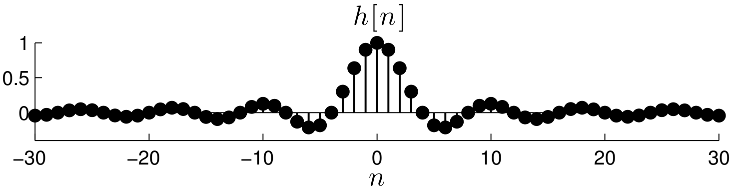
$$h[n] \;=\; \int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi} \;=\; \int_{-\omega_c}^{\omega_c} e^{j\omega n}\, \frac{d\omega}{2\pi} \;=\; \left.\frac{e^{j\omega n}}{jn}\right|_{-\omega_c}^{\omega_c} \;=\; \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn} \;=\; 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$

Impulse Response of the Ideal Lowpass Filter (2)

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] \;=\; 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



- The infamous "sinc" function!

Summary

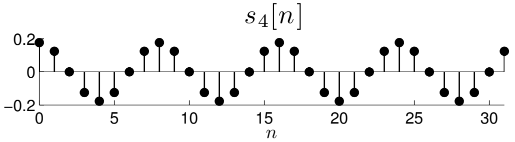
- DTFT of a rectangular pulse is a Dirichlet kernel
- DTFT of a one-sided exponential is a low-frequency bump
- Inverse DTFT of the ideal lowpass filter is a sinc function
- Work some examples on your own!

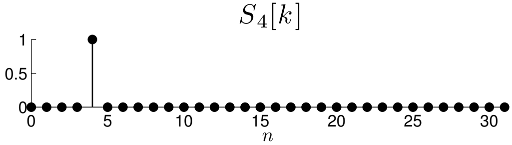
Discrete Time Fourier Transform of a Sinusoid

Discrete Fourier Transform (DFT) of a Harmonic Sinusoid

- Thanks to the orthogonality of the length-N harmonic sinusoids, it is easy to calculate the DFT of the harmonic sinusoid $x[n] = s_l[n] = e^{j\frac{2\pi}{N}ln}/\sqrt{N}$

$$X[k] \;=\; \sum_{n=0}^{N-1} s_l[n]\,\frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}} \;=\; \langle s_l, s_k\rangle \;=\; \delta[k-l]$$





- So what is the DTFT of the infinite length sinusoid $e^{j\omega_0 n}$?

DTFT of an Infinite-Length Sinusoid

- The calculation for the DTFT and infinite-length signals is much more delicate than for the DFT and finite-length signals

- Calculate the value $X(\omega_0)$ for the signal $x[n] = e^{j\omega_0 n}$

$$X(\omega_0) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega_0 n} \;=\; \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}\, e^{-j\omega_0 n} \;=\; \sum_{n=-\infty}^{\infty} 1 \;=\; \infty$$

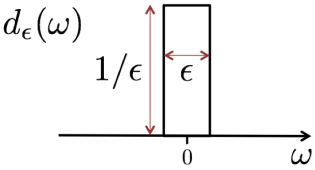
- Calculate the value $X(\omega)$ for the signal $x[n] = e^{j\omega_0 n}$ at a frequency $\omega \neq \omega_0$

$$X(\omega_0) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \;=\; \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}\, e^{-j\omega n} \;=\; \sum_{n=-\infty}^{\infty} e^{-j(\omega-\omega_0)n} \;=\; ???$$

Dirac Delta Function (1)

- One semi-rigorous way to deal with this quandary is to use the **Dirac delta "function,"** which is defined in terms of the following limit process

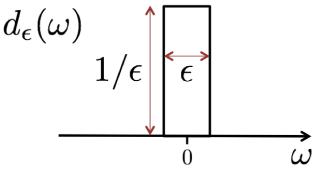
- Consider the following function $d_\epsilon(\omega)$ of the continuous variable ω



- Note that, for all values of the width ϵ, $d_\epsilon(\omega)$ always has unit area

$$\int d_\epsilon(\omega)\, d\omega \; = \; 1$$

Dirac Delta Function (2)

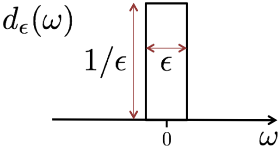


- What happens to $d_\epsilon(\omega)$ as we let $\epsilon \to 0$?
  - Clearly $d_\epsilon(\omega)$ is converging toward something that is infinitely tall and infinitely narrow but still with unit area

- The safest way to handle a function like $d_\epsilon(\omega)$ is inside an integral, like so

$$\int X(\omega)\, d_\epsilon(\omega)\, d\omega$$

Dirac Delta Function (3)



- As $\epsilon \to 0$, it seems reasonable that

$$\int X(\omega)\, d_\epsilon(\omega)\, d\omega \;\xrightarrow{\epsilon \to 0}\; X(0)$$

  and

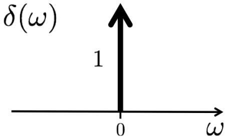
$$\int X(\omega)\, d_\epsilon(\omega - \omega_0)\, d\omega \;\xrightarrow{\epsilon \to 0}\; X(\omega_0)$$

- So we can think of $d_\epsilon(\omega)$ as a kind of "sampler" that picks out values of functions from inside an integral

- We describe the results of this limiting process (as $\epsilon \to 0$) as the **Dirac delta "function"** $\delta(\omega)$

Dirac Delta Function (4)

- **Dirac delta "function"** $\delta(\omega)$



- We write

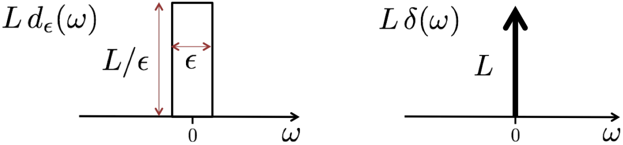
$$\int X(\omega)\,\delta(\omega)\,d\omega \;=\; X(0)$$

and

$$\int X(\omega)\,\delta(\omega-\omega_0)\,d\omega \;=\; X(\omega_0)$$

- Remarks and caveats
  - Do not confuse the Dirac delta "function" with the nicely behaved discrete delta function $\delta[n]$
  - The Dirac has lots of "delta," but it is not really a "function" in the normal sense (it can be made more rigorous using the theory of generalized functions)
  - Colloquially, engineers will describe the Dirac delta as "infinitely tall and infinitely narrow"

Scaled Dirac Delta Function



- If we scale the area of $d_\epsilon(\omega)$ by L, then it has the following effect in the limit

$$\int X(\omega)\, L\, \delta(\omega)\, d\omega \;=\; L\, X(0)$$

And Now Back to Our Regularly Scheduled Program . . .

- Back to determining the DTFT of an infinite length sinusoid

- Rather than computing the DTFT of a sinusoid using the forward DTFT, we will show that an infinite-length sinusoid is the inverse DTFT of the scaled Dirac delta function $2\pi\delta(\omega-\omega_0)$

$$\int_{-\pi}^{\pi} 2\pi\delta(\omega-\omega_0)\, e^{j\omega n}\,\frac{d\omega}{2\pi} \;=\; e^{j\omega_0 n}$$

- Thus we have the (rather bizarre) DTFT pair

$$e^{j\omega_0 n} \;\overset{\text{DTFT}}{\longleftrightarrow}\; 2\pi\,\delta(\omega-\omega_0)$$

DTFT of Real-Valued Sinusoids

- Since

$$\cos(\omega_0 n) \;=\; \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)$$

we can calculate its DTFT as

$$\cos(\omega_0 n) \;\overset{\text{DTFT}}{\longleftrightarrow}\; \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

- Since

$$\sin(\omega_0 n) \;=\; \frac{1}{2j}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right)$$

we can calculate its DTFT as

$$\sin(\omega_0 n) \;\overset{\text{DTFT}}{\longleftrightarrow}\; \frac{\pi}{j}\,\delta(\omega - \omega_0) + \frac{\pi}{j}\,\delta(\omega + \omega_0)$$

Summary

- The DTFT would be of limited utility if we could not compute the transform of an infinite-length sinusoid

- Hence, the Dirac delta “function” (or something else) is a necessary evil

- The Dirac delta has infinite energy (2-norm); but then again so does an infinite-length sinusoid

Discrete Time Fourier Transform Properties

Recall: Discrete-Time Fourier Transform (DTFT)

- Forward DTFT (Analysis)

$$X(\omega) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad -\pi \leq \omega < \pi$$

- Inverse DTFT (Synthesis)

$$x[n] \;=\; \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}, \qquad \infty < n < \infty$$

- DTFT pair

$$x[n] \;\overset{\text{DTFT}}{\longleftrightarrow}\; X(\omega)$$

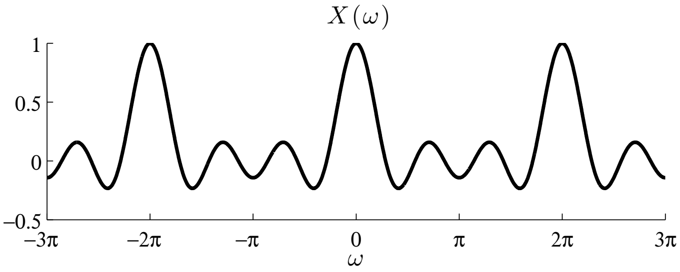
The DTFT is Periodic

- We defined the DTFT over an interval of ω of length 2π, but it can also be interpreted as **periodic** with period 2π

$$X(\omega) \;=\; X(\omega + 2\pi k), \quad k \in \mathbb{Z}$$

- Proof

$$X(\omega + 2\pi k) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi k)n} \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\, e^{-j2\pi kn} \;=\; X(\omega) \;\checkmark$$

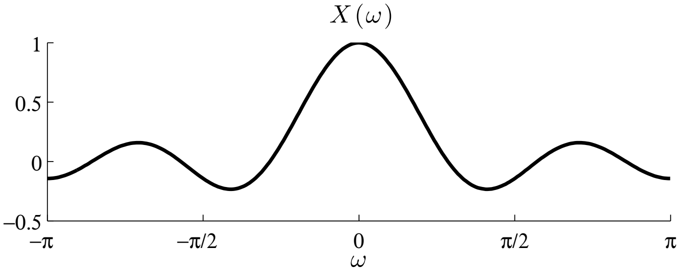


DTFT Frequencies

$$X(\omega) \; = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad -\pi \le \omega < \pi$$

- $X(\omega)$ measures the similarity between the time signal x and and a sinusoid $e^{j\omega n}$ of frequency ω

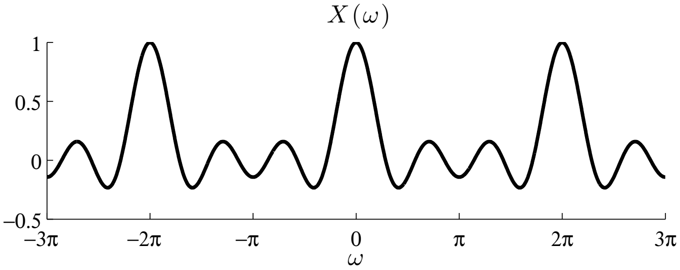
- Therefore, $X(\omega)$ measures the "frequency content" of x at frequency ω



DFT Frequencies and Periodicity

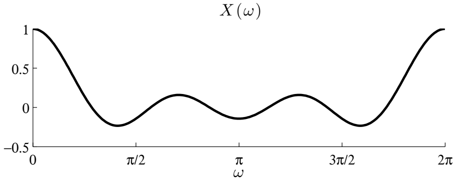
- Periodicity of DFT means we can treat frequencies mod 2π

- $X(\omega)$ measures the "frequency content" of x at frequency $(\omega)_{2\pi}$

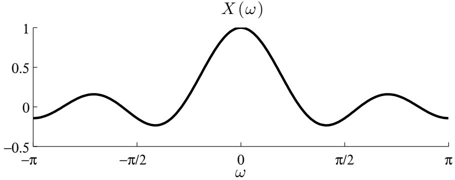


DTFT Frequency Ranges

- Periodicity of DTFT means every length-2π interval of ω carries the same information
- Typical interval 1: $0 \leq \omega < 2\pi$



- Typical interval 2: $-\pi \leq \omega < \pi$ (more intuitive)



DTFT and Time Shift

- If $x[n]$ and $X(\omega)$ are a DTFT pair then

$$x[n-m] \stackrel{\text{DFT}}{\longleftrightarrow} e^{-j\omega m} X(\omega)$$

- Proof: Use the change of variables $r = n - m$

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x[n-m]\, e^{-j\omega n} &= \sum_{r=-\infty}^{\infty} x[r]\, e^{-j\omega(r+m)} = \sum_{r=-\infty}^{\infty} x[r]\, e^{-j\omega r}\, e^{-j\omega m} \\
&= e^{-j\omega m} \sum_{r=-\infty}^{\infty} x[r]\, e^{-j\omega r} = e^{-j\omega m}\, X(\omega) \quad \checkmark
\end{aligned}$$

DTFT and Modulation

- If $x[n]$ and $X(\omega)$ are a DFT pair then

$$e^{j\omega_0 n}\,x[n] \;\overset{\text{DFT}}{\longleftrightarrow}\; X(\omega-\omega_0)$$

- Remember that the DTFT is 2π-periodic, and so we can interpret the right hand side as $X((\omega-\omega_0)_{2\pi})$

- Proof:

$$\sum_{n=-\infty}^{\infty} e^{j\omega_0 n}\,x[n]\,e^{-j\omega n} \;=\; \sum_{n=-\infty}^{\infty} x[n]\,e^{-j(\omega-\omega_0)n} \;=\; X(\omega-\omega_0) \;\checkmark$$

DTFT and Convolution

$$x \longrightarrow \boxed{h} \longrightarrow y$$

$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- If

$$x[n] \overset{\text{DTFT}}{\longleftrightarrow} X(\omega), \qquad h[n] \overset{\text{DTFT}}{\longleftrightarrow} H(\omega), \qquad y[n] \overset{\text{DTFT}}{\longleftrightarrow} Y(\omega)$$

then

$$Y(\omega) \;=\; H(\omega)\, X(\omega)$$

- Convolution in the time domain $=$ multiplication in the frequency domain

The DTFT is Linear

- It is trivial to show that if

$$x_1[n] \overset{\text{DTFT}}{\longleftrightarrow} X_1(\omega) \qquad x_2[n] \overset{\text{DTFT}}{\longleftrightarrow} X_2(\omega)$$

then

$$\alpha_1 x_1[n] + \alpha_2 x[2] \overset{\text{DFT}}{\longleftrightarrow} \alpha_1 X_1(\omega) + \alpha_2 X_2(\omega)$$

DTFT Symmetry Properties

- The sinusoids $e^{j\omega n}$ of the DTFT have symmetry properties:

$$\begin{aligned} \mathrm{Re}\left(e^{j\omega n}\right) &= \cos\left(\omega n\right) \quad \text{(even function)} \\ \mathrm{Im}\left(e^{j\omega n}\right) &= \sin\left(\omega n\right) \quad \text{(odd function)} \end{aligned}$$

- These induce corresponding symmetry properties on $X(\omega)$ around the frequency $\omega = 0$

- **Even** signal/DFT

$$x[n] = x[-n], \qquad X(\omega) = X(-\omega)$$

- **Odd** signal/DFT

$$x[n] = -x[-n], \qquad X(\omega) = -X(-\omega)$$

- Proofs of the symmetry properties are identical to the DFT case; omitted here

DFT Symmetry Properties Table

$x[n]$	$X(\omega)$	$\mathrm{Re}(X(\omega))$	$\mathrm{Im}(X(\omega))$	$\lvert X(\omega)\rvert$	$\angle X(\omega)$
real	$X(-\omega) = X(\omega)^*$	even	odd	even	odd
real & even	real & even	even	zero	even	
real & odd	imaginary & odd	zero	odd	even	
imaginary	$X(-\omega) = -X(\omega)^*$	odd	even	even	odd
imaginary & even	imaginary & even	zero	even	even	
imaginary & odd	imaginary & odd	odd	zero	even	

Summary

- DTFT is periodic with period 2π
- Convolution in time becomes multiplication in frequency
- DTFT has useful symmetry properties

Acknowledgements