edX Robo4 Mini MS – Locomotion Engineering

Week 6 – Unit 2
Raibert Vertical Hopper
Video 7.1

Segment 6.2.1
Hybrid Systems Model

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with
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July, 2017
Initial Approach to Vertical Hopper

- Model Continuous time flows
  - each mode of contact
  - governed by different VF
- Model natural guard conditions
  - physical event interrupts mode
  - locomotion: typically LO/TD
- Study/Express $mode$ map
- Model $reset$ map
- Compose
  - mode map $\circ$ reset map
  - further compose each composition in turn
- End up with return map

\[
p_{lo} = f_{PRC}^{lo}(p_{et}) \]
\[
x_{td} = f_{BF}^{td}(x_0) \]
\[
p_{bot} = f_{PRC}^{bot}(p_{td}) \]

$\mathbf{x}_{et} = \mathbf{x}_{bot} + \mathbf{x}_t$

Vertical Hopper as Hybrid System

- Physical system doesn’t “know” about modes
- But users must: e.g., control of hopping height
  - Raibert: fixed duration constant thrust
  - different duration yields different behavior

Simulation credit: Jeffrey Duperret
Vertical Hopper as Hybrid System
Reset from Flight to Compression

• Need to “hand-off” state
  ▪ from touchdown
    \[ x_{td} = \mathcal{F}_{BF}^{td}(x_0) \]
  ▪ to compression
    \[ p_{bot} = \mathcal{F}_{PRC}^{bot}(p_{td}) \]

• Use CC as reset
  \[ p_{td} = h_{PRC}(x_{td}) \]
  \[ =: \nu_{td}^{PRC}(x_{td}) \]

• Yields mode map
  \[ m_{FLT} := \nu_{td}^{PRC} \circ \mathcal{F}_{BF}^{td} \]
Reset from Compression to Decompression

- **Thrust** $\Phi_{\text{ref}}(t) \equiv \tau$
  - timed: $t \in [t_b, t_b + \Delta \tau]$
  - not event driven
  - so plays the role of a reset

- **To “hand-off” state**
  - from compression
    $$p_{bot} = f_{\text{PRC}}^{\text{bot}}(p_{td})$$
  - through thrust
    $$x_{et} = x_{bot} + x_t$$
  - to decompression
    $$p_{lo} = f_{\text{PRC}}^{\text{lo}}(p_{et})$$

- **Use appropriate CC**
  $$p_{et} = h_{\text{PRC}}(h_{\text{PRC}}^{-1}(p_{bot}) + x_t)$$
  $$=: \nu_{\text{bot}}^{\text{et}}(p_{bot})$$

- **Yields compression mode map**
  $$m_{\text{CMP}} := \nu_{\text{bot}}^{\text{et}} \circ f_{\text{PRC}}^{\text{bot}}$$
Reset from Decompression to Flight

- To reach flight mode map \( m_{\text{FLT}}(x_{lo}) := r_{\text{td}}^{\text{PRC}} \circ f_{\text{BF}}^{\text{td}}(x_{lo}) \)
- From liftoff
  \[
p_{lo} = f_{\text{PRC}}^{\text{lo}}(p_{et})
  \]
- Use CC as reset
  \[
x_{lo} = h_{\text{PRC}}^{-1}(p_{lo})
  =: r_{\text{PRC}}^{\text{lo}}(x_{td})
  \]
- Yields mode map
  \[
m_{\text{DCMP}} := r_{\text{PRC}}^{\text{lo}} \circ f_{\text{PRC}}^{\text{lo}}
  \]
Moving Ahead

• Finally have mode maps
  - express the physics
  - of each mode
  - and compose properly

• How to use them?
  - fear: infinite regress?
  - hope: what can they reveal?
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Segment 6.2.2
Hybrid Systems Model

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Where We Are Going

• Have mode maps!
• What can they reveal?
  ▪ gait stability properties
  ▪ parameter influence
  ▪ gait control affordance
• How to get there
  ▪ Insight, Data
  ▪ Models
  ▪ Analysis
  ▪ Synthesis

figures from
Steady state gait representation

- Periodic hopping orbit
  - a cycle in steady state limit
  - called limit cycle
  - “parallel” direction to flow
    - very little change
    - per flow box theorem

- Behavior summarized by
  - one dimensional section
    - “transverse:” flow cuts across
    - “return:” flow brings section back
  - no unique choice of section
    - stance bottom state
    - flight apex state
    - touchdown state
    - liftoff state
  - flow takes one section to next
    - represented by mode maps
    - each a CC between sections (uniqueness)

figure from
The (Poincare’) Return Map

• Strategy
  - fix a section (bottom)
  - define coordinates (energy, \( \rho \))
  - compose mode maps
  - to get return map

\[
\rho_{RVH} := m_{DCMP} \\
\circ m_{FLT} \quad (1) \\
\circ m_{CMP}
\]

• Study discrete dynamics

\[
\rho_{k+1} = \rho_{RVH}(\rho_k)
\]

figure from
Physical Meaning of Bottom Coordinates

• Polar RC coordinates include vertical total energy
  
  ▪ showed in Seg.2.3
  
  \[
  \rho = e_1^T p = e_1^T h_P(y) = e_1^T h_{PRC}(x)
  \]

  \[
  = \eta_{HO}(x) = \frac{1}{2}m\ddot{x}^2 + \frac{1}{2}kx^2
  \]
  
  ▪ express same equivalence using GLH parameters
  
  \[
  \rho = e_1^T p = e_1^T h_P(y) = e_1^T h_{PRC}(x)
  \]

  \[
  \rho = \eta_{GLH}(x) = \dot{x}^2 + 2\beta \dot{x}\omega x + (1 + \beta^2)\omega^2 x^2
  \]

• Total energy at bottom represents spring potential
  
  ▪ recall bottom event guard \( \gamma_{comp}(x) = \dot{x} \)
  
  ▪ hence bottom coordinates \( \rho_b = \omega^2(1 + \beta^2)x^2_b \)
  
  ▪ represent spring potential (equivalently, extension)
Decompression & Flight Mode Map Derivation

• Showed (Seg.6.1) polar RC bottom
  - since $\tilde{\gamma}_{comp}(p) = \gamma_{comp} \circ h_{PRC}^{-1}(p) = \sqrt{1 + \beta^2 \rho \sin \phi}$
  - occurs at $\phi = n\pi$ – we’ll take $n = 0$

• Show (Exrs): $y_{et} = y_b + y_t = \begin{bmatrix} \psi_b \\ \psi_t \end{bmatrix}$

  \[ \Rightarrow p_{et} = \begin{bmatrix} \rho_{et} \\ \phi_{et} \end{bmatrix} = \begin{bmatrix} \psi_t - \sqrt{\rho_b} + \frac{\psi_t^2}{\psi_t - \sqrt{\rho_b}} \\ -\pi + \arctan\left(\frac{\psi_t}{\sqrt{\rho_b - \psi_t}}\right) \end{bmatrix} \]

• Showed (Seg. 6.1+Exrs) reduce PRC VF to 1 dim

  \[ \frac{d\rho}{d\phi} = \frac{d}{dt} \rho/\frac{d}{dt} \phi = -\frac{2\beta \omega \rho}{-\omega} =: \tilde{f}_{PRC}(\rho) \]

  \[ \Rightarrow \rho_{lo} = \tilde{f}_{PRC,\phi_{et}}(\rho_{et}) = e^{2\beta(\phi_{lo} - \phi_{et})} \rho_{et} \]

• Flight is lossless (total energy conserved): $\rho_{td} = \rho_{lo}$

• Touchdown angle – use symmetry:

  $\chi_{td} = \chi_{lo} = 0 \Rightarrow \dot{\psi}_{td} = -\dot{\psi}_{lo} \Rightarrow \phi_{td} = \phi_{lo} - \pi$
Bottom Return Map Derivation

• Compression to next bottom
  ▪ via reduced PRC
    \[ \rho_{b,next} = e^{2\beta(\phi_{b,next} - \phi_{td})} \rho_{td} \]
    \[ = e^{-2\beta \phi_{lo}} \rho_{lo} \]
  ▪ and symmetry

• Compose with preceding decompression mode map

\[
\rho_{b,next} = e^{-2\beta \phi_{lo}} \rho_{lo} \\
= e^{-2\beta \phi_{lo}} e^{2\beta(\phi_{lo} - \phi_{et})} \rho_{et} \\
= e^{-2\beta \phi_{et}} \rho_{et}
\]

• Finally compose with reset from previous bottom

\[
\rho_{b,next} = \exp \left( -2\beta \left[ -\pi + \arctan \left( \frac{\psi_t}{\sqrt{\rho_b - \psi_t}} \right) \right] \right) \\
\cdot \left[ \psi_t - \sqrt{\rho_b} \right]^2 + \psi_t^2 \tag{3}
\]

\[\Rightarrow: \rho_{RVH}(\rho_b)\]
Moving Ahead

• We’ve now written out return map
• In bottom coordinates: $\rho_{k+1} = \mathcal{P}_{RVH}(\rho_k)$
  ▪ total energy at “next” bottom
  ▪ as a function of total energy at “previous” bottom
• What can it reveal?
  ▪ gait stability properties
  ▪ parameter influence
  ▪ gait control affordance
• First introduce discrete dynamical systems
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Segment 6.3.1

Return Map Analysis - Coordinates

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Raibert’s Original Hopping Analysis

• control system delivers fixed thrust each stance
  ▪ causing bouncing to come to equilibrium
  ▪ at hopping height for which energy injected
  ▪ just equals energy lost (to friction and unsprung leg mass)

• because mechanical losses are monotonic with hopping height
  ▪ a unique equilibrium hopping height
  ▪ exists for each fixed value of thrust
  ▪ and greater thrust results in greater height

paraphrase of Raibert’s ’86 MIT Press book passage from
Graphical Portrayal of Raibert’s Analysis

• Energy added and lost each stance

\[ p(\rho) \approx \rho + a(\rho) - l(\rho) \]

\[ a(\rho) := \text{energy gained} \quad (1) \]

\[ l(\rho) := \text{energy lost} \]

• Monotonic mechanical losses
  - a unique equilibrium hopping height
  - exists for each fixed value of thrust
  - and greater thrust results in greater height

• Potential complications
  - \( a \) must fall off power limits
  - might fall off “early”

figures from
D. E. Koditschek and M. Bühler
Discrete Scalar LTI System

- Exr: vertical stance hopper
  - add fixed energy, $E_b$, at bottom
  - but not enough to achieve liftoff

$$\rho_{k+1} = e^{-2\pi \beta} \rho_k + E_b =: \rho_{VSH}(\rho_k)$$ (2)

- Gives return map of the graphical form in eqn (1)
- Gives practice with discrete dynamical systems
  - LTI discrete theory mimics that of continuous time
  - stability of FP \( \rho^* = \frac{E_b}{1 - e^{-2\pi \beta}} \)
  - from eigenvalues of magnitude less than 1
Change of Coordinates to ET Section

- To simplify the return map expression
  \[ p_{RVH}(\rho) = \exp\left(-2\beta \left[-\pi + \arctan\left(\frac{\dot{\psi}_t}{\sqrt{\rho - \psi_t}}\right)\right]\right) \left[\psi_t - \sqrt{\rho}\right]^2 + \dot{\psi}_t^2 \]

- It’s convenient to rewrite in ET-coordinates
  \[ \phi = h_{bet}(\rho) := e_2^T \mathbf{r}_{bot}^{et}(\rho) = \arctan\left[\frac{\dot{y}_t}{\sqrt{\rho_b - y_t}}\right] \]
  \[ \Rightarrow h^{-1}_{bet}(\phi) = \left[\frac{\dot{y}_t}{\tan\phi} + \psi_t\right]^2 \]

- Where, recall (Seg.6.2),
  \[ \mathbf{p}_{et} = \begin{bmatrix} \rho_{et} \\ \phi_{et} \end{bmatrix} = \mathbf{r}_{bot}^{et}(\mathbf{p}_{bot}) := h_{PRC}(h^{-1}_{PRC}(\mathbf{p}_{bot})) \]
  \[ = \begin{bmatrix} \psi_t - \sqrt{\rho_b} \end{bmatrix}^2 + \dot{\psi}_t^2 \]
  \[ -\pi + \arctan\left(\frac{\dot{\psi}_t}{\sqrt{\rho_b - \psi_t}}\right) \]

figure from
D. E. Koditschek and M. Bühler,  
The (Poincare’) Return Map in ET Coords

• New ET coordinate return map representation

\[ \tilde{\rho}_{RVH}(\phi) := h_{bet} \circ \rho_{RVH} \circ h^{-1}_{bet}(\phi) = \tilde{g}_{RVH} \circ g_{RVH}(\phi) \]

\[ \tilde{g}_{RVH}(u) := \arctan \left( \frac{u}{1 + \alpha_t u} \right) ; \quad \alpha_t := \frac{\psi_t}{\dot{\psi}_t} \quad (4) \]

\[ g_{RVH}(\phi) := \sin \phi \cdot e^{\beta(\pi - \phi)} \]

• Next: study discrete dynamics \( \phi_{k+1} = \tilde{\rho}_{RVH}(\phi_k) \)
  - by finding FP
  - and their linearized dynamics

• Simpler, but not simple
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Segment 6.3.2
Return Map Analysis - Stability

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Recap: One Dimensional Discrete Dynamics

- Goal: stability of limit cycle
  - conditions for convergence
  - to isolated periodic orbit
  - from all nearby ICs
- Ingredients of Analysis
  - section
    \[ \gamma_{\text{comp}}^{-1}[0] := \{ \mathbf{x} \in \mathbb{R}^2 : \gamma_{\text{comp}}(\mathbf{x}) = 0 \} \]
    \[ \gamma_{\text{comp}}(\mathbf{x}) := \dot{\mathbf{x}} \]
    - specification using one equation
    - in two variables \( \mathbf{x} := [x, \dot{x}] \)
    - yields a one dimensional set of ICs
- Poincare’ (“return”) map

figure from
Meaning of Poincare’ (“return”) map

- What is it? Why is it helpful?
  - function specifying how
    - “next” 1 dim section crossing
    - depends upon “previous”
  - characterizing cyclic behavior
    - since value along any section
    - equivalently expresses qualitative properties
  - asymptotically (steady state)

- Choice of section
  - any “transverse” curve will do
    - e.g., liftoff $\gamma_{\text{decomp}}^{-1}[0] : = \{ x \in \mathbb{R}^2 : \gamma_{\text{decomp}}(x) = 0 \}$; $\gamma_{\text{decomp}}(x) : = \chi$
    - e.g. end-thrust points $p_{et} = h_{\text{PRC}}(h_{\text{PRC}}^{-1}(p_{bot}) + x_t) = : r_{\text{bot}}^{et}(p_{bot})$
    - all 1 dim representations
    - of the “energy” in the cycle
    - at given instant

Figure from D. E. Koditschek and M. Buhler, The International Journal of Robotics Research, 1991, op. cit.
Poincare’ Map: Bottom Coordinates

• Bottom coordinates \( \rho := \omega^2 (1 + \beta^2) \chi^2 \)
  - total energy at maximum compression
  - measures spring potential
• Poincare’ map \( \rho_{k+1} = p_{RVH}(\rho_k) \)
  - expresses next bottom energy
  - as function of previous bottom energy

\[ p_{RVH}(\rho) = \exp \left( -2\beta \left[ -\pi + \arctan \left( \frac{\dot{\psi}_t}{\sqrt{\rho} - \dot{\psi}_t} \right) \right] \right) \cdot \left( [\dot{\psi}_t - \sqrt{\rho}]^2 + \dot{\psi}_t^2 \right) \]

id(\rho) := \rho

figure from
Poincare’ Map: Bot -> ET Coordinates

- Simplify map representation
  - by CC to ET-coordinates

\[
\tilde{p}_{RVH}(\phi) := h_{bet} \circ p_{RVH} \circ h_{bet}^{-1}(\phi)
\]

(illustrated using different parameters from previous slide)

\[
id(\phi) := \phi
\]

\[
\tilde{p}_{RVH}(\phi) = \arctan \left( \frac{\dot{\psi}_t \sin \phi \cdot e^{\beta(\pi - \phi)}}{\dot{\psi}_t + \psi_t \sin \phi \cdot e^{\beta(\pi - \phi)}} \right)
\]

figure from
First Step of Analysis: Fixed Points

- Study discrete dynamics $\phi_{k+1} = \tilde{P}_{RVH}(\phi_k)$
  - by finding FP $\tilde{P}_{RVH}(\phi^*) = \phi^* \iff \phi^* \in \{\phi_0^* := 0, \phi_1^* > 0\}$
  - and studying their linearized dynamics

- ET coordinate map can be factored
  $\tilde{P}_{RVH}(\phi) = \tilde{g}_{RVH} \circ g_{RVH}(\phi)$

  $\tilde{g}_{RVH}(u) := \arctan \left( \frac{u}{1 + \alpha_t u} \right); \quad \alpha_t := \frac{\psi_t}{\dot{\psi}_t}$

  $g_{RVH}(\phi) := \sin \phi \cdot e^{\beta(\pi - \phi)}$

- Quickly see 0 must be FP: it’s FP for both factors

- More work to ascertain existence of $\phi_1^* > 0$

Second Step: Linearization at $\phi_0^* = 0$

- Calculus
  \[ \tilde{P}_{RVH}(\phi) = D\tilde{p}_{RVH}(\phi) = \tilde{g'}_{RVH}\big|_{u=g_{RVH}(\phi)} \cdot g'_{RVH}(\phi) \]
  \[ \tilde{g'}_{RVH}(u) = \frac{1}{u^2 + (1 + \alpha_t u)^2} \]
  \[ g'_{RVH}(\phi) = e^{\beta(\pi - \phi)} (\cos \phi - \beta \sin \phi) \]

- Implies
  \[ \tilde{P}_{RVH}(0) = \tilde{g'}_{RVH}\big|_{0=g_{RVH}(0)} \cdot g'_{RVH}(0) \]
  \[ = 1 \cdot e^{\beta\pi - 0} (\cos 0 - \beta \sin 0) \]
  \[ = e^{\beta\pi} > 1 \]

- Hence FP at 0 is unstable
  - hopping is pumped up
  - from very low energy states
Third Step: Linearization at $\phi_1^* > 0$

- Introduce quotient map on the set $\phi > 0$
  
  \[ q(\phi) := \frac{\tilde{p}_{RVH}(\phi)}{\phi} \]

  - know $q(\phi) > 1$ for $0 < \phi < \phi_1^*$
  - since 0 is unstable FP
  - and $\phi_1^*$ is unique FP on $\phi > 0$

- Know $q(\phi_1^*) = 1$ so $q'(\phi_1^*) < 0$
  - now use calculus

  \[ 0 > q'(\phi_1^*) = \frac{1}{\phi_1^*} [\tilde{p}'_{RVH}(\phi_1^*) - 1] \]

  \[ \Rightarrow 1 > \tilde{p}'_{RVH}(\phi_1^*) \]

- More arguments show $\tilde{p}'_{RVH}(\phi_1^*) > -1$


- So linearization is asymptotically stable

\[ 1 > \tilde{P}_{RVH}(\phi_1^*) := D\tilde{p}_{RVH}(\phi_1^*) > -1 \]
Summary and Conclusion

- ET-coordinate representation of Poincare’ map
  - has two FP \( \tilde{p}_{RVH}(\phi^*) = \phi^* \iff \phi^* \in \{\phi_0^* := 0, \phi_1^* > 0\} \)
  - whose linearized dynamics
    - is unstable at \( \phi_0^* \): \( \tilde{P}_{RVH}(\phi_0^*) = e^{\beta \pi} > 1 \)
    - and asymptotically stable at \( \phi_1^* \): \( |\tilde{P}_{RVH}(\phi_1^*)| < 1 \)
- Conjugation preserves FP and linearized eigenvalues
- Hence stance energy map has same properties
  - two FP
    \( p_{RVH}(\rho^*) = \rho^* \iff \rho^* \in \{\rho_0^* := h^{-1}_{bet}(\phi_0^*) = 0, \rho_1^* := h^{-1}_{bet}(\phi_1^*) > 0\} \)
    - same stability properties \( P_{RVH}(\rho^*) = \tilde{P}_{RVH}(\phi^*) \)
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Segment 6.3.3

Return Map Analysis - Conclusion

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What Has Been Shown?

- Used Hooke’s law stance model of hopper
  - to show that constant thrust pumps energy
  - with a unique locally asymptotically stable limit cycle
- Additional arguments using this model give
  - essentially global basin for unique limit cycle
  - with possible “hunting” (negative slope linearization)
  - but preclude “limping” (period-two FP )
    - occurs only in physically meaningless parameter regime
    - wherein end-thrust resets directly into flight mode
Verified Much of Raibert’s Original Analysis

• control system delivers fixed thrust each stance
  ▪ causing bouncing to come to equilibrium
  ▪ at hopping height for which energy injected
  ▪ just equals energy lost (to friction)

• although mechanical losses may not be monotonic
  ▪ a unique equilibrium hopping height
  ▪ exists for each fixed value of thrust
  ▪ and greater thrust results in greater height

• however poorly chosen parameters
  ▪ may result in “hunting” (oscillatory convergence to FP)
  ▪ and closely related nonlinear model exhibits “limping”
Limping: When can it Happen?

- Simulations of pneumatic spring model
  - in physically plausible regime
    - where compression force from high prior apex
    - back-drives the pneumatic pressure chamber
  - exhibit robust “limping”
    - convergence to alternation between
    - identically repeated
    - high-long & short-low hops

- Poincare’ map analysis
  - reveals FP-destabilizing bifurcation
  - to asymptotically stable period-two orbit

- Raibert reported empirical “limping”
  - (personal comm.)
  - but seemed due to higher dof effects

figures from
D. E. Koditschek and M. Bühler, 
The RVH as Dynamical Template

- Approximate RVH Poincare’ Map
- Is “anchored” in Buehler’s juggler
  - system settles down
  - to purely vertical orbits
  - whose return maps
  - have steady state properties
  - as predicted
- Introduce more formal notion soon

figures from: Property of Penn Engineering and Daniel E. Koditschek

Bifurcation Studies with Buehler’s Juggler

- Bifurcation
  - qualitative change
  - in attractor structure
  - due to systematic parameter adjustment

- Extensive theory available

Textbook reference:

figures from:
Dynamical Systems Thinking in Robotics

• We’ve encoded our tasks as dynamical attractors
• Whose basins function as abstract symbols
  ▪ regions of state space
  ▪ wherein the task is guaranteeably programmed
  ▪ and indefatigably achieved

• To be “composed”
  ▪ achieving some more complicated behavior
  ▪ from some simpler, well understood components

• Dynamical Systems Theory
  ▪ gives mathematically tractable
  ▪ physically robust and achievable
  ▪ predictive and composable symbols