Solutions to Final Exam
Principles of Economics with Calculus
Caltech/edX
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Problem 1: Solution

The consumer's problem is:

$$\max_{x \geq 0} \ln x - 2\sqrt{x}$$

The FOC for this problem is

$$\frac{1}{x} - \frac{1}{\sqrt{x}} = 0$$

Solving the FOC for x gives the solution $x^* = 1$

Note that the FOC in this problem is necessary and sufficient. To see this, first let $T(x) = B(x) - C(x)$ where $B(x) = \ln(x)$ and $C(x) = 2\sqrt{x}$. Now, note that $B' > 0$ and $B'' \leq 0$, and $C' > 0$ and $C'' \geq 0$. Additionally, $B' \to 0$ as $x \to \infty$. As a result of the concavity and crossing conditions being satisfied, x^* is the unique maximum.

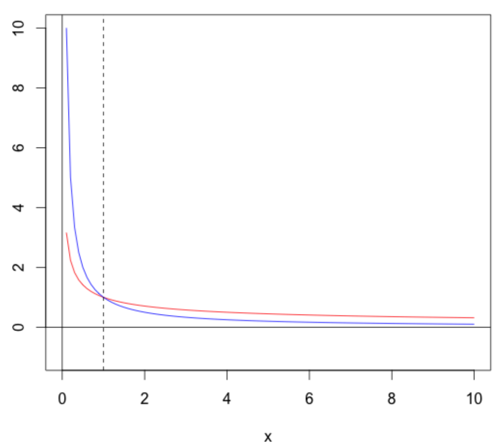

Figure 1: The marginal benefit curve, shown in blue, and the marginal cost curve, shown in red satisfy the necessary and sufficient conditions such that the solution, x^* is the unique maximum.

Problem 2: Solution

First, let's solve for the amount of good a produced in equilibrium. Note that a consumer of type X gets a benefit of

$$B_x(x_a) = \ln x_a + 100 \ln y_b$$

and the consumer of type X is the only type that demands good a.
To find the quantity produced in equilibrium, we set marginal benefit equal to marginal cost. So, we see that for each consumer of type X

$$\frac{1}{x_a} = p_a \implies x_a = \frac{1}{p_a} \implies x_a^* = 1$$

In the above, we know that $p_a = 1$ in equilibrium by looking at the firm's problem and setting the marginal cost of production for good a, 1, equal to p_a
Then, since there are 100 consumers, the total x_a, and hence good a, produced is $100x_a^* = 100$

Next, let's solve for the amount of good b produced in equilibrium. Note that a consumer of type Y gets a benefit of

$$B_y(y_b) = \ln y_b + 100 \ln x_a$$

and the consumer of type Y is the only type that demands good b.
Again, to find the quantity produced in equilibrium, we set marginal benefit equal to marginal cost. So, we find that for each consumer of type Y

$$\frac{1}{y_b} = p_b \implies y_b = \frac{1}{p_b} \implies y_b^* = 2$$

In the above, we know that $p_b = \frac{1}{2}$ in equilibrium by looking at the firm's problem and setting the marginal cost of production for good b, $\frac{1}{2}$, equal to p_b
Then, since there are 100 consumers, the total y_b, and hence good b, produced is $100y_b^* = 200$

Next, let's find the Pareto Optimal allocations. To do this, we must maximize the total social benefit, which is

$$100B_x(x_a) + 100B_y(y_b) = 10100 \ln x_a + 10100 \ln y_b$$

Taking the marginal total social benefit of x_a and setting equal to the marginal social cost of production, we find

$$\frac{10100}{x_a} = 1 \implies x_a^{PO} = 10100$$

which is the Pareto optimal allocation of good a. Likewise for good b,

$$\frac{10100}{y_b} = \frac{1}{2} \implies y_b^{PO} = 20200$$

which is the Pareto optimal allocation of good b.

Problem 3: Solution

First let's write how the subsidy changes the consumer's problem. Under the subsidy, a consumer of type X will have benefit,

$$B_x(x_a) = \ln x_a + 100 \ln y_b + \alpha x_a$$

and a consumer of type Y will have benefit,

$$B_y(y_b) = \ln y_b + 100 \ln x_a + \beta y_b$$

Including the subsidy when setting marginal benefit equal to marginal cost, which the subsidy has not affected, for a consumer of type X reveals

$$\frac{1}{x_a} + \alpha = 1 \implies x_a^* = \frac{1}{1-\alpha}$$

Since there are 100 consumers, the total x_a demanded is $100x_a^* = \frac{100}{1-\alpha}$. In order to restore optimality, the amount of good a demanded must equal the Pareto Optimal amount of good a from Problem 2. Solving this,

$$\frac{100}{1-\alpha} = 10100 \implies \alpha = \frac{100}{101}$$

To find β, setting marginal benefit equal to marginal cost for a consumer of type Y reveals

$$\frac{1}{y_b} + \beta = \frac{1}{2} \implies y_b^* = \frac{1}{\frac{1}{2}-\beta}$$

Since there are 100 consumers, the total y_b demanded is $100y_b^* = \frac{100}{\frac{1}{2}-\beta}$. In order to restore optimality, the amount of good b demanded, $\frac{100}{\frac{1}{2}-\beta}$, should equal the Pareto Optimal amount of good b from Problem 2. So,

$$\frac{100}{\frac{1}{2}-\beta} = 20200 \implies \beta = \frac{50}{101}$$

Problem 4: Solution

A rational individual faces the problem

$$\max_{q\in[0,4]} 2\sqrt{q} - q$$

Taking the first order condition for the above,

$$\frac{1}{\sqrt{q}} - 1 = 0 \implies q^* = 1$$

So the rational individual buys 1 unit of q.

Next, let's examine the irrational individual, with $B > 1$. This individual makes decisions by solving

$$\max_{q\in[0,4]} \frac{2}{3}q^{\frac{3}{2}} + Bq - q$$

The FOC for this problem is

$$\sqrt{q} + B - 1 = 0$$

Note, when $B > 1$ the FOC will not be satisfied for any value of q and the SOC condition is $\frac{1}{2\sqrt{q}} > 0$ which suggests the object we wish to maximize is convex. The marginal benefit is $MB(q) = \sqrt{q} + B$ and the marginal cost is $MC(q) = 1$ when $q \in [0,4]$. Now, because the marginal benefit always exceeds the marginal cost, the area under the marginal benefit curve will always be larger than the area under the marginal cost curve, and hence, the irrational individual will consume as much as possible. Since the maximum possible consumption is 4, $q^I = 4$

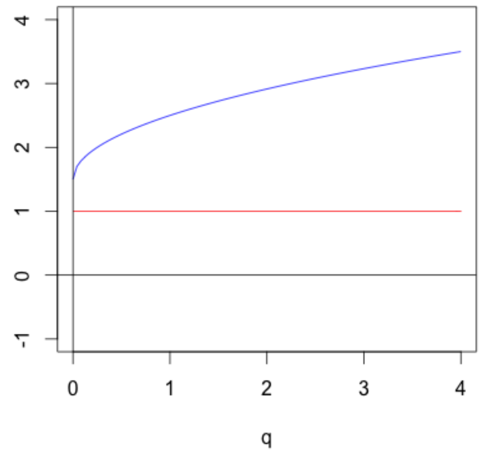

Figure 2: The marginal benefit curve, shown in blue, is always above the marginal cost curve, depicted in red

Finally, let's compute the maximum value of B at which the irrational indivudal buys nothing. Note that the individual will not buy anything when the area under the marginal benefit curve is less than or equal to the area under the marginal cost curve. Hence, we can solve the following equation

$$\int_0^4 (\sqrt{q} + B)dq \leq \int_0^4 1dq \implies B \leq -\frac{1}{3}$$

So, when $B \leq -\frac{1}{3}$ then the individual will not purchase anything.

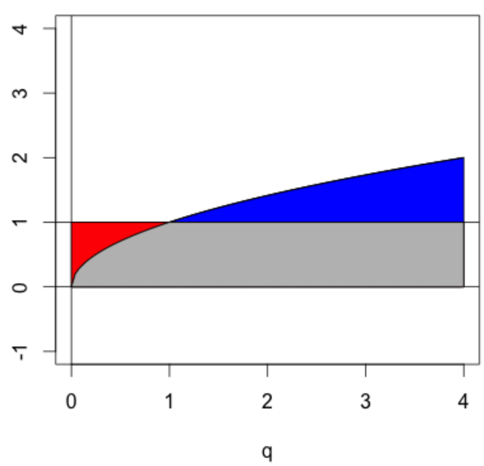

Figure 3: The individual will not purchase anything when the area under the marginal benefit curve (the blue plus grey area), is less than or equal to the area under the marginal cost curve (the red plus grey area)

Problem 5: Solution

The consumer surplus for the rational individual with benefit function $B(x) = 2\sqrt{x}$ is given by

$$B(q^*) - B(0) - pq^* = B(1) - B(0) - p = 2 - 0 - 1 = 1$$

Finding the consumer surplus for the irrational individual with $B > 1$, which is still computed with the rational benefit function, is given by

$$B(q^I) - B(0) - pq^I = B(4) - B(0) - 4p = 4 - 0 - 4 = 0$$

Finally, the consumer surplus for the irrational individual with $B = -10$ will be 0 as the consumer does not purchase any q for such a value of B, as seen in Problem 4.

Problem 6: Solution

When $B = 1$, the benefit function for the irrational consumer is $\hat{B}(x) = \frac{2}{3}q^{\frac{3}{2}} + q$. Additionally, as a result of the per-unit tax, the consumer now faces the following problem

$$\max_{q\in[0,4]} \frac{2}{3}q^{\frac{3}{2}} - \tau q$$

which has FOC is $\sqrt{q} - \tau = 0$ with a SOC of $\frac{1}{2\sqrt{q}} > 0$. With the subsidy, the consumer has $MB(q) = \sqrt{q} + 1$ and $MC(q) = 1 + \tau$. Note that just as in Problem 4, the consumer will choose to purchase nothing when the area under under the marginal benefit curve is less than the area under the marginal cost curve. Writing down the equation for this,

$$\int_0^4 (\sqrt{q} + 1)dq \leq \int_0^4 (1 + \tau)dq \implies \tau \geq \frac{4}{3}$$

Hence, the minimal τ that induces the consumer to purchase nothing is $\frac{4}{3}$. Furthermore, note that when $\tau > \frac{4}{3}$, the individual will choose to purchase nothing and have a consumer surplus of 0. When $\tau < \frac{4}{3}$, the individual will choose to purchase as much as possible, and will buy 4 units, implying the consumer surplus will be 0.

Problem 7: Solution

Since there are 3 firms, this is a market with oligopolistic competition.
First, note that we can rewrite aggregate demand to get the inverse aggregate demand, i.e.

$$X^D(p) = \frac{9000}{p} \implies p^D(q) = \frac{9000}{q}$$

Next, recall that the problem of each firm is

$$\max_{q\geq 0} q(\frac{9000}{q + 2\bar{q}}) - \frac{q^2}{20}$$

where $\bar{q}$ is the production of either of the other two firms. Now, the first order condition for the above problem is

$$\frac{9000(2\bar{q})}{(q + 2\bar{q})^2} - \frac{q}{10} = 0$$

As all three firms have the same cost function, we know that $q = \bar{q}$, and solving the above FOC reveals $q^* = 100\sqrt{2}$. Since there are 3 firms, the total amount produced is

$$3q^* = 300\sqrt{2} \approx 424.26$$

Finally, we plug aggregate production back into the inverse aggregate demand to solve for the equilibrium price

$$p^D(q^*) = \frac{9000}{300\sqrt{2}} \approx 21.21$$

Problem 8: Solution

First we solve for the market equilibrium under perfect competition. Aggregate demand is given as $X^D(p) = \frac{9000}{p}$. To find aggregate supply, set marginal cost equal to price to find a firm's supply, then sum across all firms. So,

$$c_i'(q_i) = p \implies p = \frac{q_i}{10} \implies x_i^S(p) = 10p \implies X^S(p) = 30p$$

Next, set aggregate demand equal to aggregate supply to find the equilibrium market price.

$$30p = \frac{9000}{p} \implies p^* = \sqrt{300} \approx 17.32$$

Plugging p^* back into X^S or X^D finds the equilibrium quantity, which is $30\sqrt{300} \approx 519.62$ Comparing the perfect competition outcome to the outcome in the Problem 7, we can compute the deadweight loss, which is given by

$$\int_{424.26}^{519.62} (\frac{9000}{q} - \frac{q}{30})dq \approx 324.62$$

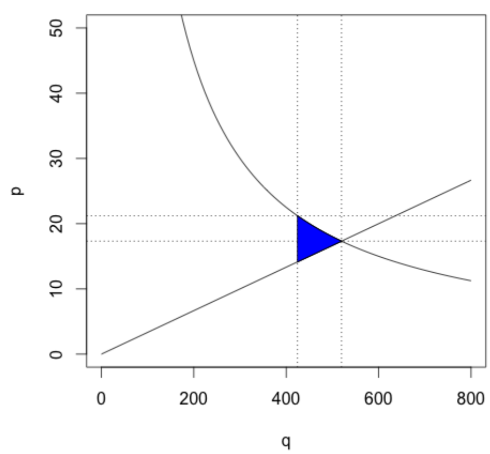

Figure 4: The deadweight loss is the shaded area in blue. The values of q range over the equilibrium quantity differences between oligopolistic competition and perfect competition