# Unit 3: Producer Theory

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# 1 Model of the firm

## 1.1 Key properties of the model

- Key assumption: firms maximize profits subject to
  - Technological constraints: natural limits to production, given existing technology
  - Economic constraints: limits driven by markets
- Simplifications:
  1. Static model: no time
  2. No uncertainty about how actions map to profits
  3. No innovation
  4. No managers

## 1.2 Primer on partial derivatives

- Let $F(x, y)$ be a function of two variables
- Define:

$$\frac{\partial F}{\partial x} = \text{ marginal change } dF \text{ induced by a marginal change } dx\text{, holding } y \text{ constant.}$$

$$\frac{\partial F}{\partial y} = \text{ marginal change } dF \text{ induced by a marginal change } dy\text{, holding } x \text{ constant.}$$

- In practice,
$$\frac{\partial F}{\partial x} = \frac{d}{dx} F(x, y),$$
treating $y$ as a constant; and
$$\frac{\partial F}{\partial y} = \frac{d}{dy} F(x, y),$$
treating $x$ as a constant.

- The mechanics are otherwise the same as the univariate case.

## 1.3 Technological constraints

- Production technology takes inputs, produces outputs
- For this course, assume two inputs:
  - $k$ = capital: machines, buildings, etc.
  - $l$ = labor: number of workers (or number of total employee hours)
- Production function: $F(k, l)$
- *isoquants* = level sets of the production function
- Important concepts:
  - Marginal product of capital, $MPK = \frac{\partial F}{\partial k}$
  - Marginal product of labor, $MPL = \frac{\partial F}{\partial l}$
- Basic properties of $F(k, l)$
  - Productive inputs: $MPL, MPK > 0$
  - Eventually decreasing returns to scale:
    – $\frac{\partial}{\partial k} MPK < 0$ for sufficiently large $k$
    – $\frac{\partial}{\partial l} MPL < 0$ for sufficiently large $l$
    – This implies that there is an optimal scale of production

- Taxonomy of production functions:
  - CRS: constant returns to scale: for all $\lambda > 0$, for all $(k, l)$, $F(\lambda k, \lambda l) = \lambda F(k, l)$
  - DRS: decreasing returns to scale: for all $\lambda > 1$, for all $(k, l)$, $F(\lambda k, \lambda l) < \lambda F(k, l)$
  - IRS: increasing returns to scale: for all $\lambda > 1$, for all $(k, l)$, $F(\lambda k, \lambda l) > \lambda F(k, l)$
  - Important: These are global properties that need to be satisfied at every $(k, l)$
  - Important: some functions are neither CRS, DRS or IRS
- Example: Cobb-Douglas production function (used a lot in applied economics)
  - $F(k, l) = Ak^{\alpha}l^{\beta}, \alpha, \beta > 0, A > 0$
  - $A =$ total factor productivity
  - $\alpha + \beta = 1 \implies$ CRS
  - $\alpha + \beta < 1 \implies$ DRS
  - $\alpha + \beta > 1 \implies$ IRS
  - Proof for $\alpha + \beta = 1$ :

$$\begin{aligned} F(\lambda k, \lambda l) &= A(\lambda k)^{\alpha}(\lambda l)^{\beta} \\ &= A\lambda^{\alpha+\beta}k^{\alpha}l^{\beta} \\ &= \lambda Ak^{\alpha}l^{\beta} \\ &= \lambda F(k, l) \qquad \square \end{aligned}$$

- Minimal production scale = minimal level of inputs $(\bar{k}, \bar{l})$ below which no output can be produced.
  - Simple example:

$$F(l) = \begin{cases} 0 & \text{if } l < \bar{l} \\ \beta(l - \bar{l}) & \text{otherwise} \end{cases}$$

  - Leads to increasing returns like behavior in the production function

## 1.4 Decision problem of the competitive firm

- Economic constraints for competitive firm: price takers for inputs and outputs
  - Output sold at price $p$
  - Inputs bought at $r$ per unit capital and $w$ per unit labor
- Firm's problem:

$$\max_{q,k,l\geq 0} qp - (rk + wl)$$

$$\text{s.t. } q = F(k, l)$$

- Solution:

| | |
|---|---|
| $q^*(p, w, r)$ | Supply |
| $k^*(p, w, r)$ | Demand for capital |
| $l^*(p, w, r)$ | Demand for labor |

- Remarks:
  1. Firm operates in three markets
  2. Supply and input demands each depend on three variables $(p, w, r)$
  3. Can substitute equality constraint to make a two-variable optimization problem (see below)
- Two equivalent ways of studying the problem of a competitive firm:
  - Choice over inputs:

$$\max_{k,l\geq 0} pF(k, l) - (rk + wl)$$

    Solution is $k^*(p, w, r), l^*(p, w, r)$. From this we get $q^*(p, w, r) = F(k^*, l^*)$
  - Choice over output:
    * Step 1: solve

$$\min_{k,l} rk + wl$$

$$\text{s.t. } F(k, l) = q$$

      to get $k^{min}, l^{min}$. Define $C(q|w, l) = rk^{min}(q|w, r) + wl^{min}(q|w, r)$

* Step 2: solve

$$\max_{q \geq 0} pq - C(q|w, r)$$

Solution is $q^*(p, w, r)$. From this we get $l^*(p, w, r) = l^{min}(q^*|w, r)$ and $k^*(p, w, r) = k^{min}(q^*|w, r)$

- IMPORTANT: Profit maximization requires cost minization
  - Profit maximized at $q^*, k^*, l^* \implies k^*, l^*$ minimize cost of producing $q^*$ given input prices $w, r$
    * Proof: suppose there exist $\hat{k}, \hat{l}$ such that
      1. $q^* = F(\hat{k}, \hat{l})$
      2. $r\hat{k} + w\hat{l} < rk^* + wl^*$

      Then $q^*, \hat{k}, \hat{l}$ feasible and yields higher profit, which contradicts that profits are maximized at $q^*, k^*, l^*$

# 2 Cost functions

## 2.1 Cost minimization problem

- Graphically: cost minimized when isocost tangent to isoquant with desired level of production
  - Isoquant: $\frac{\partial F}{\partial k} dk + \frac{\partial F}{\partial l} dl = 0$
    Slope of isoquant:

$$\frac{dl}{dk} = -\frac{\frac{\partial F}{\partial k}}{\frac{\partial F}{\partial l}}$$

  - Isocost: $l = \frac{\bar{c}}{w} - \frac{r}{w} k$
    Slope of isocost:

$$-\frac{r}{w}$$

  - So $l^{min}, k^{min}$ satisfy:
    1. $q = F(l^{min}, k^{min})$
    2. $\frac{\frac{\partial F}{\partial k}}{\frac{\partial F}{\partial l}} = \frac{r}{w}$

- Intuition
  - $\frac{\frac{\partial F}{\partial k}}{\frac{\partial F}{\partial l}} =$ Marginal Rate of Technical Substitution (MRTS) = If we increase capital, by how much can we decrease labor and still produce the same amount?
  - $\frac{r}{w}$ = relative prices = If we increase capital, by how much must we decrease labor to keep costs unchanged?
  - If these are unequal (at an interior point), there is a cheaper way to produce the same quantity
- REMARK: Theoretically, can have MRTS $\neq \frac{r}{w}$ at optimum, if it's a corner solution. Rare in practice, though.

## 2.2 Example

- Cobb-Douglas prod. function: $F(k,l) = Ak^\alpha l^\beta, \alpha, \beta > 0, \alpha + \beta = 1$
- For this case:

$$MPK = A\alpha k^{\alpha-1} l^\beta = \frac{\alpha q}{k}$$

$$MPL = A\beta k^\alpha l^{\beta-1} = \frac{\beta q}{l}$$

- Solution of the cost minimization problem must be interior, since output is zero along axes
- Cost minimization solution given by conditions:
  1. $q = Ak^\alpha l^\beta$
  2. $\frac{MPK}{MPL} = \frac{r}{w} \sim \frac{\alpha l}{k\beta} = \frac{r}{w} \sim l = \frac{r}{w}\frac{\beta}{\alpha}k$
- Substitute 2 into 1:

$$q = Ak^\alpha \left[\frac{r}{w}\frac{\beta}{\alpha}k\right]^\beta \implies \begin{cases} k^{min} = \frac{q}{A}\left[\frac{w}{r}\right]^\beta \left[\frac{\alpha}{\beta}\right]^\beta \\ l^{min} = \frac{q}{A}\left[\frac{r}{w}\right]^\alpha \left[\frac{\beta}{\alpha}\right]^\alpha \end{cases}$$

- From this, it follows that:

$$C(q|w,r) = rk^{min} + wl^{min} = q\frac{r^{\alpha}w^{\beta}}{A}\left[\left(\frac{\alpha}{\beta}\right)^{\beta} + \left(\frac{\beta}{\alpha}\right)^{\alpha}\right]$$

- Remarks:

  1. $C$ linear in $q$, this is related to fact that underlying production function satisfies CRS
  2. $C$ is not linear in $r$ or $w$ alone, since can substitute $k$ and $l$ when relative prices change
  3. $C$ is linear in $r$ and $w$ together; e.g. if they both double, $C$ doubles.

## 2.3 Properties of cost functions

- Basic definitions

  - VC = variable costs: depend on level of production $q$
  - SFC = semi-fixed costs: kick in if $q > 0$, but don't depend on $q$ otherwise
  - FC = fixed costs: must be paid even if $q = 0$, don't depend on $q$ at all
  - TC($q$) = total cost = FC + SFC($q$) + VC($q$)
  - MC($q$) = marginal cost = $\frac{d}{dq}TC(q) = \frac{d}{dq}VC(q)$
  - AVC($q$) = average variable cost = $\frac{VC(q)}{q}$
  - ATC($q$) = average total cost = $\frac{TC(q)}{q} = AVC(q) + \frac{FC+SFC(q)}{q}$

- NOTE: For any $q > 0$, $\frac{d}{dq}SFC(q) = 0$

- Example 1: CRS

$$TC(q) = \mu q, \mu > 0$$

- Example 2: CRS w/ SFC

$$TC(q) = SFC + \mu q$$

with $SFC, \mu > 0$. In this case, although MC constant, ATC decreases w/ $q$

- Example 3: DRS w/ SFC

$$TC(q) = SFC + \mu q^2$$

with $SFC, \mu > 0$. In this case, ATC crosses MC at ATC's minimum

- Example 4: Semi-DRS w/ SFC

$$TC(q) = SFC + 1000\sqrt{q} + q^3$$

with $SFC > 0$. Let $q^{min}$ be the point at which MCs are minimal. Then, for $q < q^{min}$ the cost function exhibits IRS, and for $q > q^{min}$ it exhibits DRS.

- Useful properties for DRS, semi-DRS, semi-DRS+SFC cost functions:

  - Property 1:
  $$\begin{aligned} q < q^{ATC-min} &\implies MC < ATC \\ q = q^{ATC-min} &\implies MC = ATC \\ q > q^{ATC-min} &\implies MC > ATC \end{aligned}$$
  - Property 2:
  $$\begin{aligned} q < q^{AVC-min} &\implies MC < AVC \\ q = q^{AVC-min} &\implies MC = AVC \\ q > q^{AVC-min} &\implies MC > AVC \end{aligned}$$
  - Property 3:
  $$q^{MC-min} \leq q^{AVC-min} \leq q^{ATC-min}$$

## 2.4 Application: Optimal production scale

- Example 1:
  - How should $Q$ units be produced in a society, if firms all have same technology, which is DRS w/ SFC?
  - Need to find optimal number of firms, $n$, and optimal quantity for each firm to produce, $q_1, \ldots, q_n$
  - Solution:

$$n \approx \frac{Q}{q^{ATC-min}}$$

$$q_1 = \cdots = q_n \approx q^{ATC-min}$$

- Example 2:
  - Same question, but technology has CRS and SFC.
  - Solution:

$$n = 1$$

$$q_1 = Q$$

## 2.5 Application: Bookstores and technological change

- Before e-commerce: $q^{ATC-min}$ low, so $n$ large; i.e. optimal to have many bookstores
- After e-commerce: very large SFC, but very low MC, so ATC decreases indefinitely, so $n = 1$
- E-commerce reduces overall costs, but leaves very, very few bookstores
- From point of view of pure productive efficiency this is good
- But can't say yet whether good or bad overall for consumers, since:
  - Markets only work well when there are many firms
  - Some people may get some utility from having bookstores around
- We will revisit these issues in later units

# 3 Supply functions

## 3.1 Basics

- Firm's profit maximization problem:

$$\max_{q \geq 0} pq - c(q)$$

  - Solution $q^*(p)$ is the supply function
  - Recall: $c$ also depends on $w, r$; therefore so does $q^*$
  - Supply function gives firm's optimal output as a function of ouput price, given the cost function associated with the input prices

- Pure DRS
  - $c' > 0, c'' > 0 \implies$ concavity conditions hold, and $c'(0) = 0 < p$.
  - So solution satisfies $MR = MC$, i.e. $p = MC$
  - $q^*(p)$ is same locus of points as MC(q)

- DRS with SFCs
  - Shut down condition: If $p < minATC$, then $q^*(p) = 0$
  - If $p \geq minATC$, then $q^*(p)$ given by FOC $p = MC(q)$.

- DRS with SFCs and FCs
  - $ATC^{SFC} = AVC + \frac{SFC}{q}$
  - $ATC^{SFC+FC} = AVC + \frac{SFC+FC}{q}$
  - Solution similar to the case when $FC = 0$, since FC doesn't affect behavior
  - But the key curve for determining when production becomes positive is the ATC-SFC curve
  - But note that the FCs affect profits

- CRS

- $MC = ATC = AVC = \mu$
- In this case we have that:

$$q^*(p) = \begin{cases} 0 & p < \mu \\ \text{anything} & p = \mu \\ \infty & p > \mu \end{cases}$$

- Pure semi-DRS:
  - $FC = SFC = 0$
  - In this case we have that:

$$q^*(p) = \begin{cases} 0 & p < minATC \\ q \text{ s.t. } p = MC(q) \& \ p \geq ATC(q) & \text{otherwise} \end{cases}$$

## 3.2 Example

- $TC = FC + SFC + \frac{1}{2}q^2$, with $FC = SFC = 9$
- $MC = q$
- $ATC^{SFC} = \frac{9}{q} + \frac{q}{2}$
- $minATC - SFC$ at $q = 3\sqrt{2}$
- Thus:

$$q^*(p) = \begin{cases} 0 & \text{if } p \leq 3\sqrt{2} \\ p & \text{if } p \geq 3\sqrt{2} \end{cases}$$

## 3.3 Example

- Consider case of per-unit tax
- $\theta$ : tax $\tau > 0$ per unit produced and sold
- Now, problem is

$$\max_{q \geq 0} pq - [c(q) + \tau q]$$

- $MC_{\tau>0} = MC_{\tau=0} + \tau$

- $ATC_{\tau>0} = ATC_{\tau=0} + \tau$
- Solution:
$$q^*_{\tau>0}(p) = q^*_{\tau=0}(p-\tau)$$

### 3.4 Example

- Consider a sell-one-donate-one mandatory policy: firm must donate one item to government for each item sold
- $TC = SFC + \frac{1}{2}q^2$, with $SFC = 8$
- Now, problem is
$$\max_{q\geq 0} pq - \left[SFC + \frac{1}{2}(2q)^2\right],$$
where $q$ denotes the amount sold in the market
- $MR = p$
- $MC = 4q$
- $ATC = \frac{8}{q} + 2q$
- $q^{minATC} = 2$, $minATC = 8$
- So
$$q^*(p) = \begin{cases} 0 & p \leq 8 \\ \frac{p}{4} & p \geq 8 \end{cases}$$

## 4 Producer surlus

- Producer surplus $= PS(\theta) = \Pi(\theta) - \Pi_{no-trade}$, with $\Pi_{no-trade} = -FC$
- Example: Free-trade at price $p$
  - $PS(\theta) = pq^* - [SFC(q^*) + VC(q^*)]$
  - Let $\bar{p} = minATC^{SFC}$.
  - Let denote $p^*(q)$, the inverse supply function.
  - If $p < \bar{p}$, $PS(p) = 0$.

- Consider the case with $p \geq \bar{p}$, where the supply equals the MC above the level $\bar{q}$
- We get

$$PS(\theta|p \geq \bar{p}) = (p - \bar{p})\bar{q} + \int_{\bar{q}}^{q^*(p)} (p - p^*(q))\, dq$$

- Note that PS is only a function entirely of observables variable!

- Example: per-unit production tax
  - $\theta$ : per unit tax $\tau$ and price $p$
  - In this case $PS(\theta) = pq^*(\theta) - [SFC + VC + \tau q^*(p)]$
  - Thus,

$$PS(\theta) = \begin{cases} 0 & \text{if } p < \bar{p} + \tau \\ (p - (\bar{p} + \tau))\bar{q} + \int_{\bar{q}}^{q^*(p)} (p - \tau - c'(q))\, dq & \text{otherwise} \end{cases}$$

  - Now, $c'(q)$ not observable, but $p^*_\tau(q) = \tau + c'(q)$ is, so we still have PS as function of observables

# 5 Final remarks

- KEY RESULT 1:
  - Cost minimization problem:

$$c(q|w, r) = \min_{k,l \geq 0} rk + wl$$

$$\text{s.t. } q = F(k, l)$$

  - Solution $k^{min}(q|w, l), l^{min}(q|w, l)$ satisfy:

1. $q = F(k^{min}, l^{min})$
2. $\frac{\frac{\partial F(k^{min}, l^{min})}{\partial k}}{\frac{\partial F(k^{min}, l^{min})}{\partial l}} = \frac{r}{w}$

- KEY RESULT 2:
  - Properties of supply functions for DRS, semi-DRS, or semi-DRS+SFC under free trade
  - Profit maximization problem $\Pi(p) = \max_{q \geq 0} pq - c(q)$
  - Solution:

$$q^*(p) = \begin{cases} 0 & p < minATC - SFC \\ q \text{ s.t. } p = MC(q) \& \ p \geq ATC(q) & \text{otherwise} \end{cases}$$

  - Tip/reminder: check shut-down conditions!

- KEY RESULT 3: Producer surplus is given by

$$\begin{aligned} PS(\theta) &= \Pi(\theta) - \Pi_{no-trade} \\ &= revenue(\theta) - (SFC + VC(\theta)) \end{aligned}$$