Video 7.1
Vijay Kumar

# Control of Affine Systems

**State**

$$x \in \mathbb{R}^n$$

**Input**

$$u \in \mathbb{R}^m$$

**State equations**

$$\dot{x} = f(x) + g(x)u$$

**Output**

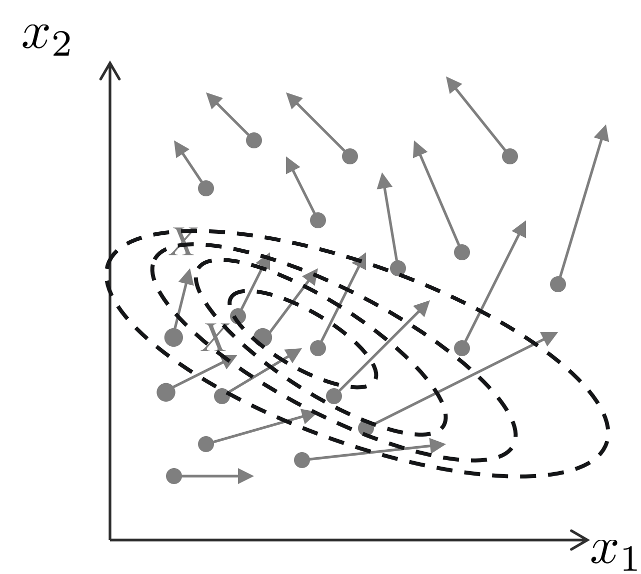
$$y \in \mathbb{R}^m$$

$$y = h(x)$$

# Lie Derivative

Function

$$f : \mathbb{R}^n \to R$$




Vector Field

$$X(x) = \begin{bmatrix} X_1(x) \\ X_2(x) \\ \dots \\ X_n(x) \end{bmatrix}$$

Lie derivative of $f$ along $X$

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \dots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

$$\mathcal{L}_X f = X \cdot \nabla f$$

$$\begin{bmatrix} X_1 \; X_2 \; \dots \; X_n \end{bmatrix}$$

# Lie Derivative

Lie derivative of $f$ along $X$

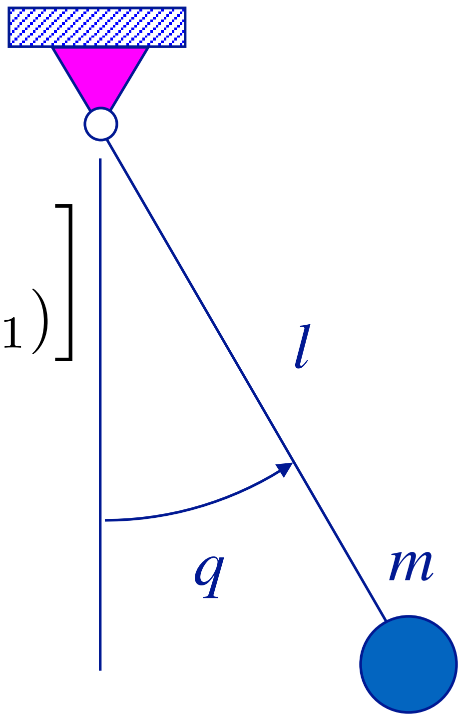
$$\mathcal{L}_X f = \begin{bmatrix} X_1 & X_2 & \dots & X_n \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \dots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Example: $n$=2

$$X = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{bmatrix}$$

$$f = -l\cos x_1$$




$$\mathcal{L}_X f = \begin{bmatrix} x_2 & -\frac{g}{l}\sin x_1 \end{bmatrix} \begin{bmatrix} l\sin x_1 \\ 0 \end{bmatrix}$$

$$\boxed{x_2 l \sin x_1}$$

# Example: Controlling a Single Output

**Output**

$$y \in \mathbb{R}$$

Want

$$\dot{y} - \dot{y}^{\text{des}} + k(y - y^{\text{des}}) = 0$$

or

$$\ddot{y} - \ddot{y}^{\text{des}} + k_1(\dot{y} - \dot{y}^{\text{des}}) + k_2(y - y^{\text{des}}) = 0$$

Need derivative of the output function

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}\left(f(x) + g(x)u\right)$$

Lie Derivatives

$$\mathcal{L}_f h = \frac{\partial h}{\partial x} f(x) \qquad \mathcal{L}_g h = \frac{\partial h}{\partial x} g(x)$$

# Single Input, Single Output, First Order Dynamics

State equations

$$\dot{x} = f(x) + g(x)u$$

Output

$$y = h(x)$$

Rate of change of output

$$\dot{y} = \mathcal{L}_f h + (\mathcal{L}_g h)\, u$$

Control law $\boxed{\text{if } \mathcal{L}_g h \neq 0}$

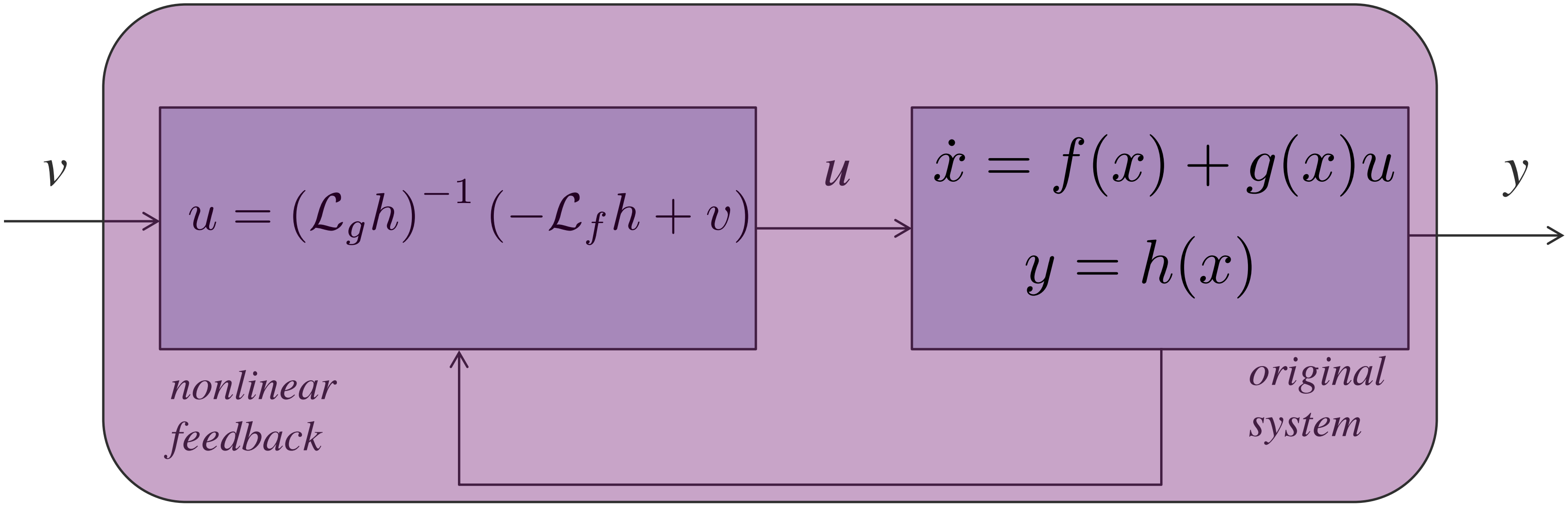
$$u = \frac{1}{\mathcal{L}_g h}\left(-\mathcal{L}_f h + \dot{y}^{\text{des}} + k(y^{\text{des}} - y)\right)$$

Closed loop system behavior

*Error exponentially converges to zero*

$$\dot{y} - \dot{y}^{\text{des}} + k(y - y^{\text{des}}) = 0$$

# Input Output Linearization




*new system* $\dot{y} = v$

*Nonlinear feedback transforms the original nonlinear system to a new linear system*

*Linearization is exact (distinct from linear approximations to nonlinear systems)*

# Affine, Single Input Single Output

State $x$ $\quad x \in \mathbb{R}^n$

Input $u$ $\quad u \in \mathbb{R}$

State equations $\quad \dot{x} = f(x) + g(x)u$

Output $\quad y = h(x) \in \mathbb{R}$

Rate of change of output

$$\dot{y} = \mathcal{L}_f h + (\mathcal{L}_g h)\, u$$

Control law

if $\mathcal{L}_g h \neq 0$

$$u = \frac{1}{\mathcal{L}_g h}\left(-\mathcal{L}_f h + \dot{y}^{\mathrm{des}} + k(y^{\mathrm{des}} - y)\right)$$

if $\mathcal{L}_g h = 0$ $\quad \dot{y} = \mathcal{L}_f h$ (rate of change of output is independent of $u$)

Explore higher order derivatives of output

*nonzero?*

$$\ddot{y} = \mathcal{L}_f \mathcal{L}_f h + \boxed{(\mathcal{L}_g \mathcal{L}_f h)}\, u$$

# Affine, Single Input Single Output

State $x$ $\quad x \in \mathbb{R}^n$

Input $u$ $\quad u \in \mathbb{R}$

State equations $\quad \dot{x} = f(x) + g(x)u$

Output $\quad y = h(x) \in \mathbb{R}$

Rate of change of output

$$\dot{y} = \mathcal{L}_f h + (\mathcal{L}_g h)\, u$$

Control law

if $\mathcal{L}_g h \neq 0$

$$u = \frac{1}{\mathcal{L}_g h}\left(-\mathcal{L}_f h + \dot{y}^{\mathrm{des}} + k(y^{\mathrm{des}} - y)\right)$$

if $\mathcal{L}_g h = 0$ $\quad \dot{y} = \mathcal{L}_f h$ (rate of change of output is independent of $u$)

if $(\mathcal{L}_g \mathcal{L}_f h) \neq 0$

$$u = \frac{1}{\mathcal{L}_g \mathcal{L}_f h}\left(-\mathcal{L}_f \mathcal{L}_f h + \ddot{y}^{\mathrm{des}} + k_1(\dot{y}^{\mathrm{des}} - \dot{y}) + k_2(y^{\mathrm{des}} - y)\right)$$

# Affine, Single Input Single Output

State $x$ $\qquad x \in \mathbb{R}^n$

Input $u$ $\qquad u \in \mathbb{R}$

State equations $\qquad \dot{x} = f(x) + g(x)u$

Output $\qquad y = h(x) \in \mathbb{R}$

Rate of change of output

$$\dot{y} = \mathcal{L}_f h + (\mathcal{L}_g h)\, u$$

Control law

$$u = \frac{1}{\mathcal{L}_g \mathcal{L}_f h} \left( -\mathcal{L}_f \mathcal{L}_f h + \ddot{y}^{\text{des}} + k_1(\dot{y}^{\text{des}} - \dot{y}) + k_2(y^{\text{des}} - y) \right)$$

Closed loop system behavior

$$\ddot{y} - \ddot{y}^{\text{des}} + k_1(\dot{y} - \dot{y}^{\text{des}}) + k_2(y - y^{\text{des}}) = 0$$

*Error exponentially converges to zero*

# Affine, Single Input Single Output

State $x$

$$x \in \mathbb{R}^n$$

Input $u$

$$u \in \mathbb{R}$$

State equations

$$\dot{x} = f(x) + g(x)u$$

Output

$$y = h(x) \in \mathbb{R}$$

$$\mathcal{L}_f^2 h = \mathcal{L}_f \left(\mathcal{L}_f h\right)$$

$$\mathcal{L}_f^3 h = \mathcal{L}_f \left(\mathcal{L}_f \left(\mathcal{L}_f h\right)\right)$$

$$\cdots$$

Relative degree, $r$ *The index of the first nonzero term in the sequence*

$$\mathcal{L}_g h,\ \mathcal{L}_g \mathcal{L}_f h,\ \mathcal{L}_g \mathcal{L}_f^2 h, \ldots, \mathcal{L}_g \mathcal{L}_f^k h, \ldots$$

$r = k + 1$

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# Video 7.2
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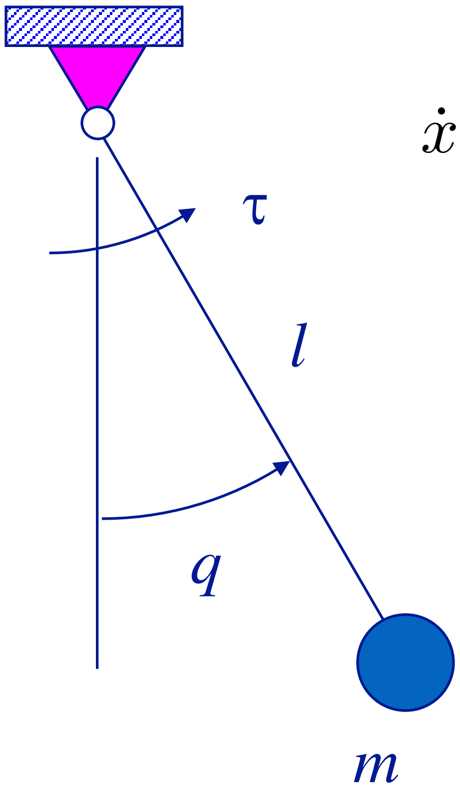
# Example 1. Single degree of freedom arm




$$ml^2\ddot{q} + \frac{1}{2}mgl\sin q = \tau$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

$$\dot{x} = \underbrace{\boxed{\begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{bmatrix}}}_{f(x)} + \underbrace{\boxed{\begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}}}_{g(x)} u$$

$$h = x_1$$

$$\mathcal{L}_g h = 0$$

$$\mathcal{L}_f h = x_2$$

$$\boxed{\mathcal{L}_g \mathcal{L}_f h = \frac{1}{ml^2}}$$

*r*=2

$$\mathcal{L}_f^2 h = -\frac{g}{l}\sin x_1$$

# Affine, SISO

*Linear control, model independent*

*feed forward*

*feedback*

$r$=1

$$u = \boxed{\frac{1}{\mathcal{L}_g h}} \left(-\mathcal{L}_f h + \boxed{\dot{y}^{\text{des}} + k(y^{\text{des}} - y)}\right)$$

$r$=2

$$u = \boxed{\frac{1}{\mathcal{L}_g \mathcal{L}_f h}} \left(-\mathcal{L}_f \mathcal{L}_f h + \boxed{\ddot{y}^{\text{des}} + k_1(\dot{y}^{\text{des}} - \dot{y}) + k_2(y^{\text{des}} - y)}\right)$$

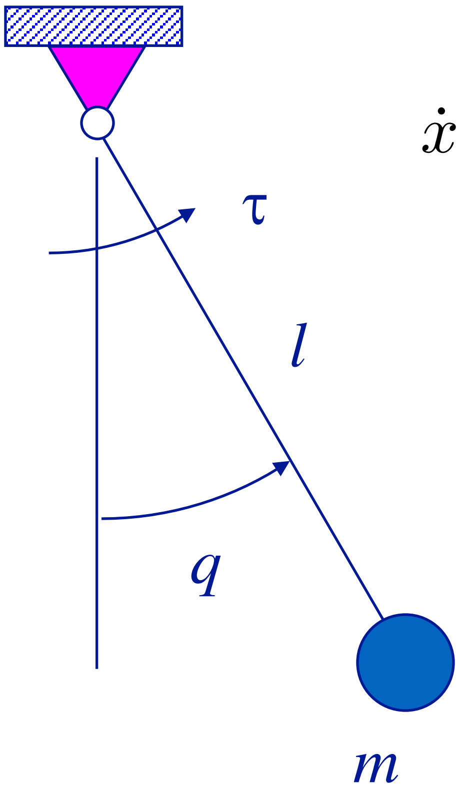
$r$=3

$$u = \boxed{\frac{1}{\mathcal{L}_g \mathcal{L}_f^2 h}} \left(-\mathcal{L}_f^3 h + \boxed{\dddot{y}^{\text{des}} + k_1(\ddot{y}^{\text{des}} - \ddot{y}) + k_2(\dot{y}^{\text{des}} - \dot{y}) + k_3(y^{\text{des}} - y)}\right)$$

*General form of control law*

$$u = \alpha(x) + \boxed{\beta(x)}\boxed{v}$$

# Single degree of freedom arm



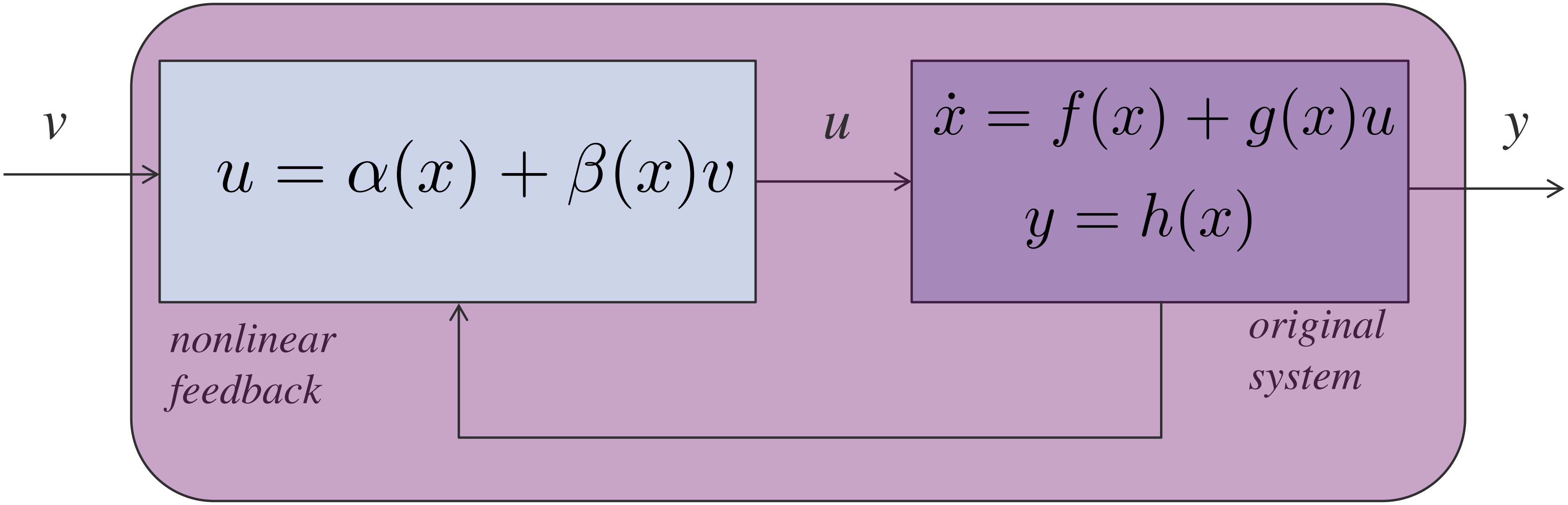

$$ml^2\ddot{q} + \frac{1}{2}mgl\sin q = \tau \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

$$\dot{x} = \underbrace{\boxed{\begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{bmatrix}}}_{f(x)} + \underbrace{\boxed{\begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}}}_{g(x)} u \qquad h = x_1$$

$$\mathcal{L}_g\mathcal{L}_f h = \frac{1}{ml^2} \qquad \mathcal{L}_f^2 h = -\frac{g}{l}\sin x_1$$

$$\boxed{u = \frac{1}{\mathcal{L}_g\mathcal{L}_f h}\left(-\mathcal{L}_f\mathcal{L}_f h + \ddot{y}^{\mathrm{des}} + k_1(\dot{y}^{\mathrm{des}} - \dot{y}) + k_2(y^{\mathrm{des}} - y)\right)}$$

# Input Output Linearization
# Single Input, Single Output, Relative degree *r*




*new system* $y^{(r)} = v$

*Nonlinear feedback transforms the original nonlinear system to a new linear system*

*Linearization is exact (distinct from linear approximations to nonlinear systems)*

# Multiple Input Multiple Output Systems

State $x$ $\quad x \in \mathbb{R}^n$

Input $u$ $\quad u \in \mathbb{R}^m$

$$\dot{x} = \underset{n \times 1}{f(x)} + \underset{n \times m}{g(x)}u$$

Output $\quad y = h(x) \in \mathbb{R}^m$

Assume each output has relative degree $r$

Nonlinear feedback law

$$u = \left(\mathcal{L}_g \mathcal{L}_f^{r-1} h\right)^{-1}_{m \times m} \left(-\mathcal{L}_f^r h + v\right)$$

leads to the equivalent system

$$\underset{m \times 1}{y^{(r)}} = \underset{m \times 1}{v}$$

# Fully-actuated robot arm (*n* joints, *n* actuators)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

Dynamic model

- $M$ is the positive definite, $n$ by $n$ inertia matrix
- $C(q,\dot{q})\dot{q}$ is the $n$-dimensional vector of Coriolis and centripetal forces
- $N$ is the $n$-dimensional vector of gravitational forces
- $\tau$ is the $n$-dimensional vector of actuator forces and torques

# Fully-actuated robot arm (continued)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

$$u = \tau \in \mathbb{R}^n$$

$$\dot{x} = \begin{bmatrix} x_2 \\ -M(x_1)^{-1}(N(x_1) + C(x_1, x_2)x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ M(x_1)^{-1} \end{bmatrix} u$$

$$y = q \in \mathbb{R}^n$$

# Fully-actuated robot arm (continued)

$$f(x) = \begin{bmatrix} x_2 \\ -M(x_1)^{-1}(N(x_1) + C(x_1, x_2)x_2) \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ M(x_1)^{-1} \end{bmatrix}$$

$$h(x) = x_1$$

$$\mathcal{L}_g h = 0, \ \mathcal{L}_g \mathcal{L}_f h \neq 0$$

Relative degree is 2

$$u = \boxed{(\mathcal{L}_g \mathcal{L}_f h)^{-1}} \left( \boxed{-\mathcal{L}_f \mathcal{L}_f h} + \boxed{\ddot{y}^{\text{des}} + k_1(\dot{y}^{\text{des}} - \dot{y}) + k_2(y^{\text{des}} - y)} \right)$$

$$M(x_1)$$

$$\boxed{-M(x_1)^{-1}(N(x_1) + C(x_1, x_2)x_2)}$$

Control law

$$u = (C(x_1, x_2)x_2 + N(x_1)) + \boxed{M(x_1)} \boxed{(\ddot{y}^{des} + k_1(\dot{y}^{des} - \dot{y}) + k_2(y^{des} - y))}$$

*Method of computed torque* (Paul, 1972)

*Inverse dynamics approach to control* (Spong *et al*, 1972)

# Under Actuated Systems

*The number of inputs is smaller than the number of degrees of freedom!*

# Kinematic planar cart

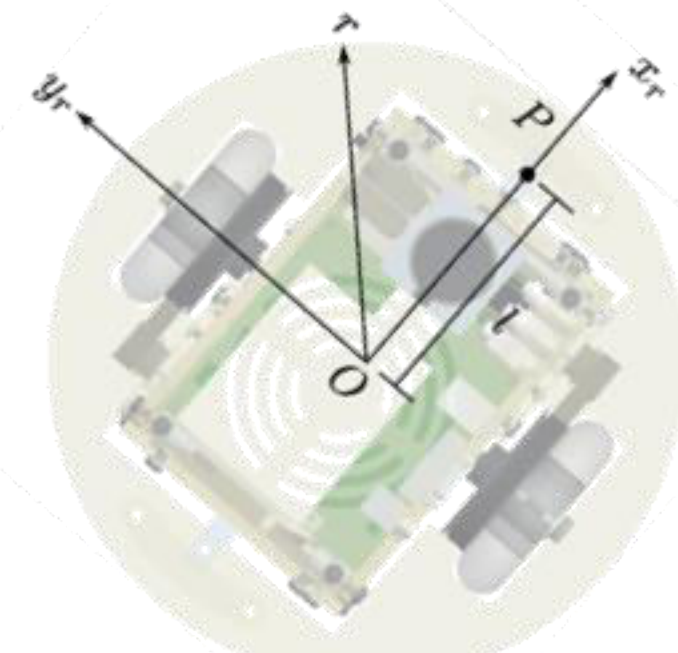
State equations, inputs

*2 inputs, 3 degrees of freedom*

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = \omega$$

$$\dot{X} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$\dot{X} = g(X)u$$




Outputs

$$\begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} x + L\cos\theta \\ y + L\sin\theta \end{bmatrix}$$

$$y = h(x) = \begin{bmatrix} x + L\cos\theta \\ y + L\sin\theta \end{bmatrix}$$

Relative degree is 1

$$\dot{y} = \begin{bmatrix} \cos\theta & -L\sin\theta \\ \sin\theta & L\cos\theta \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

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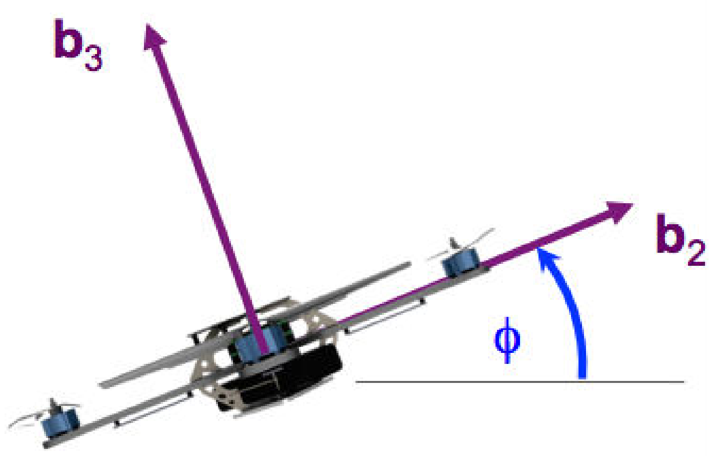
# Planar Quadrotor




$$\begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

*2 inputs, 3 degrees of freedom*

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ z \\ \phi \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{\phi} \\ 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

# Three-Dimensional Quadrotor