Data Structures and Algorithms (10)

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Higher Education Press, 2008.6 (the "Eleventh Five-Year" national planning textbook)

https://courses.edx.org/courses/PekingX/04830050x/2T2014/

Chapter 10 Search

- 10.1 Search in a linear list
- 10.2 Search in a set
- 10.3 Search in a hash table
- Summary

Basis Concepts

- Search

  The process of finding a record with its key value equal to a given value in a set of records, or the records whose keys meet some specific criteria.

- The efficiency of search is very important
  - Especially for big data
  - Need special storage processing for data

Methods of Improving Search Efficiency

- Sorting
- Indexing
- Hashing

- Take much time
- preprocessing (finished before search)
- disadvantages :
  - Unsuitable for range searches
  - Generally, duplicate keys are not allowed

- Organize the data into a table
- Get the position of records in the table according to the key values

- Make the most of auxiliary index information
- Sacrifice space
- To improve search efficiency

- When hashing is not suitable for disk-oriented applications, we can use B trees.

Average Search Length (ASL)

- Comparison of keys: main operation of search
- Average Search Length
  - Average number of comparisons during search
  - The time metric for evaluating search algorithms

$$ASL = \sum_{i=1}^{n} P_i C_i$$

- P_i is probability of searching the *i-th element*
- C_i is the number of comparisons needed to find the i-th element

Other Metrics for Evaluating Search Algorithms

- Considerations when evaluating search algorithms
  - The storage needed
  - Implementation difficulties
  - ...

Thinking

- Assume that a linear list is (a, b, c), and the probabilities of searching a, b, c are 0.4, 0.1, 0.5 respectively
  - What is the ASL of sequential search algorithms? (which means how many times of comparisons of key values are needed to find the specific element on the average)

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Search in a Linear List

- 10.1.1 Sequential search
- 10.1.2 Binary search
- 10.1.3 Blocking search

Sequential Search

- Compare the key values of records in a linear list with the given value one by one
  - If the key value of a record is equal to the given value, the search hits;
  - Otherwise the search misses (cannot find the given value in the end)
- Storage: sequential or linked
- Sorting requirements: none

Sequential Search with Sentinel

```
// Return position of the element if hit; otherwise return 0
template  <class Type>
class Item {
private:
    Type  key;                              // key field
                                            // other fields
public:
    Item(Type value):key(value) {}
    Type getKey() {return key;}             // get the key
    void setKey(Type k){ key=k;}            // set the key
};
vector<Item<Type>*>  dataList;
template  <class Type>  int SeqSearch(vector<Item<Type>*>& dataList, int
length, Type k) {
    int i=length;
    dataList[0]->setKey (k);                // set the 0th element as the element
                                            to be searched, set the lookout
    while(dataList[i]->getKey()!=k) i--;
    return i;                               // return the position of the element
}
```

Performance Analysis of the Sequential Search

- Search hits: assume the probability of searching any key value is uniform: $P_i = 1/n$

$$\sum_{i=0}^{n-1} P_i \cdot (n-i) = \frac{1}{n}\sum_{i=0}^{n-1}(n-i)$$

$$= \frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}$$

- Search misses: assume that n+1 times of comparisons are needed when the search misses (with a sentinel)

Average Search Length of Sequential Search

- Assume the probability of search hit is p, and the probability of search miss is q=(1-p)

$$\begin{aligned}\mathrm{ASL} &= p\cdot\frac{n+1}{2}+q\cdot(n+1)\\ &= p\cdot\frac{n+1}{2}+(1-p)(n+1)\\ &= (n+1)(1-p/2)\end{aligned}$$

- $(n+1)/2 < ASL < (n+1)$

Pros and Cons of Sequential Search

- Pros: insertion in $\Theta(1)$ time
  - We can insert a new element into the tail of list
- Cons: search in $\Theta(n)$ time
  - Too time-consuming

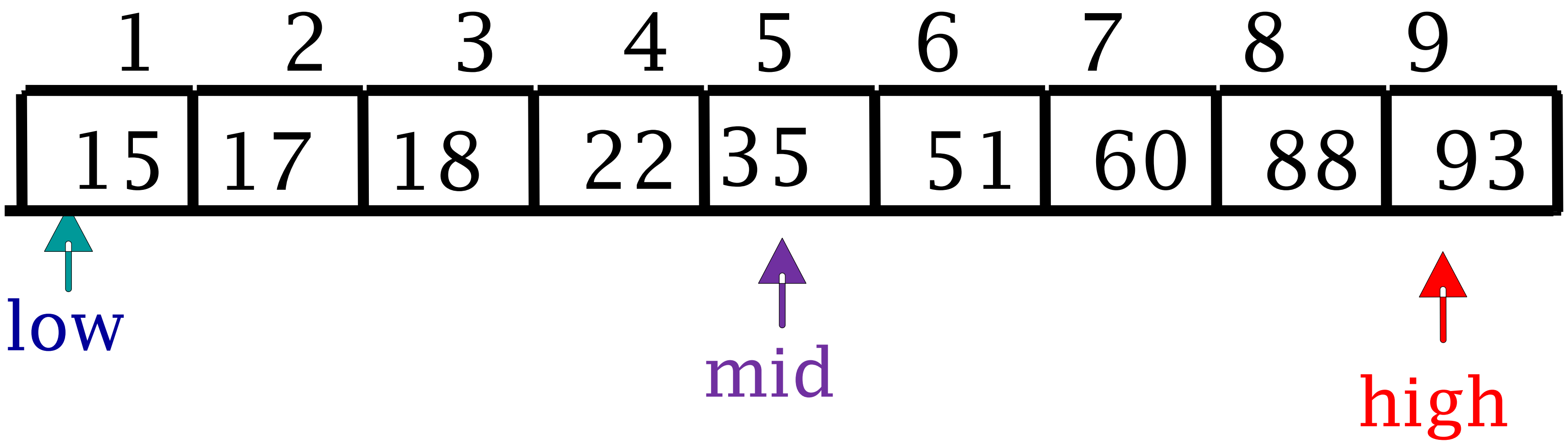
Binary Search

- Compare any element dataList[i].Key with the given value K, there are three situations:
  - (1) Key = K, succeed, return dataList[i]
  - (2) Key > K, the element to find must be before dataList[i] if exists
  - (3) Key < K, the element to find must be after dataList[i] if exists
- Reduce the range of latter search

Binary Search Algorithm

```
template  <class Type> int BinSearch (vector<Item<Type>*>& dataList, int
length, Type k){
    int low=1, high=length, mid;
    while (low<=high)  {
        mid=(low+high)/2;
        if (k<dataList[mid]->getKey())
            high = mid-1;                // drop the right half of the search range
        else if (k>dataList[mid]->getKey())
            low = mid+1;                 // drop the left half of the search range
       else return mid;                  // return if succeeds
    }
    return 0;                            // if fails, return 0
}
```

Key value 18 low=1 high=9



The first time: l=1, h=9, mid=5; array[5]=35>18

The second time: l=1, h=4, mid=2; array[2]=17<18

The third time: l=3, h=4, mid=3; array[3]=18 = 18

Performance Analysis of the Binary Search

- Maximum search length is

$$\lceil \log_2(n+1) \rceil$$

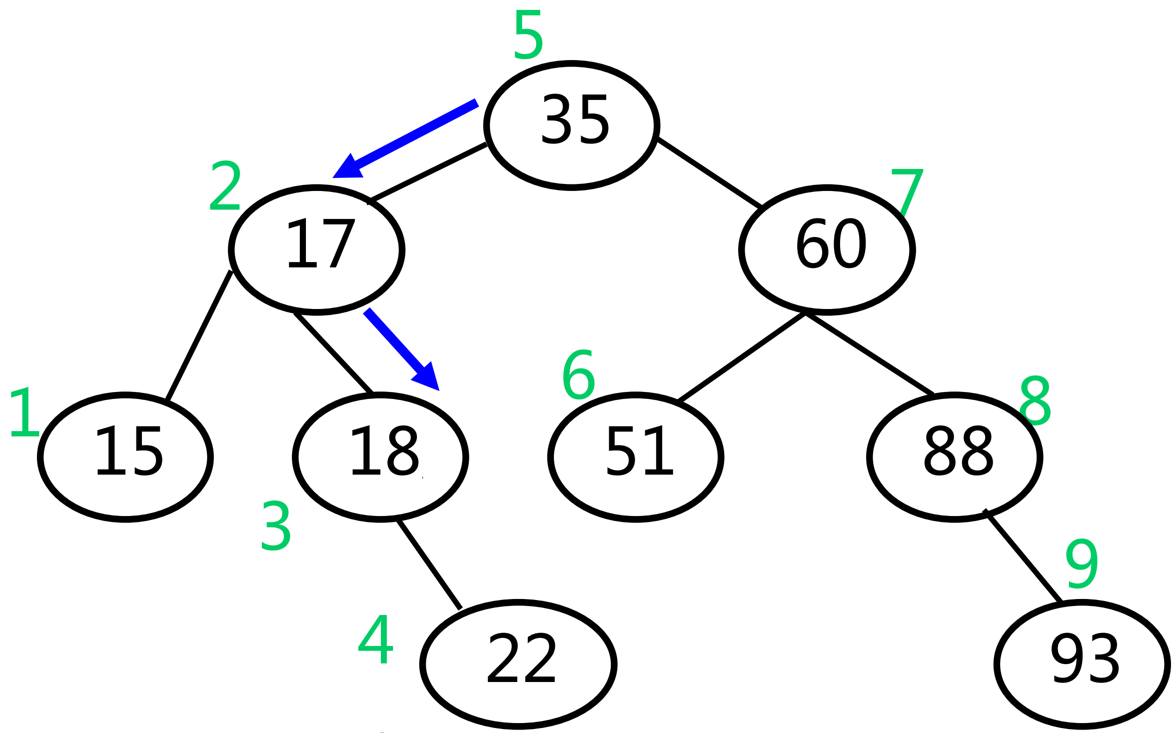
- Failed search length is

$$\lceil \log_2(n+1) \rceil$$

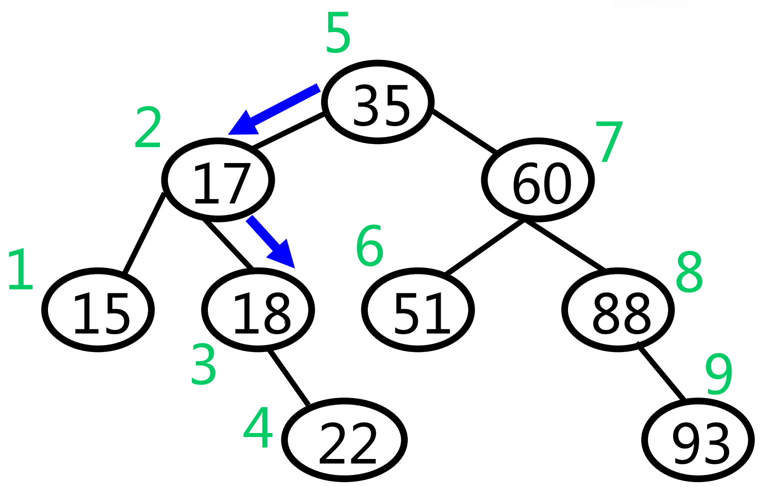
Or

$$\lfloor \log_2(n+1) \rfloor$$

- In the complexity analysis
  - The logarithm base is 2
  - When the log base changes, the order of complexity will not change



Performance Analysis of the Binary Search



- ASL of successful search is:

$$ASL = \frac{1}{n}\left(\sum_{i=1}^{j} i \cdot 2^{i-1}\right)$$

$$= \frac{n+1}{n}\log_2(n+1) - 1$$

$$\approx \log_2(n+1) - 1 \qquad (n > 50)$$

- Pros: the average and maximum search length is in the same order, and the search is very fast
- Cons: need sorting, sequential storage, difficult to update (insertion/deletion)

Ideas of the Blocking Search

- "Ordering between blocks"
  - Assume that the linear list contains *n* data element, split it into *b* blocks
  - The maximum element in any block must be smaller than the minimum element in the next block
  - Keys of elements are not always ordered in one block
- Tradeoff between sequential and binary searches
  - Not only fast
  - But also enables flexible update

Blocking Search – Index Sequential Structure

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
22	12	13	9	8		33	42	44	24	48		60	80	74	49	86	53

- Link: starting position of a block
- Key: Maximum key value in the block
- Count: #elements in a block

link:	0	6	12
Key:	22	48	86
count:	5	5	6

Performance Analysis of Blocking Search

- Blocking search is a two-level search
  - First, find the block where the specific element stays at, with ASL_b
  - Second, find the specific element inside that block, with ASL_w

$$ASL = ASL_b + ASL_w$$
$$\approx \log_2 (b+1)-1 + (s+1)/2$$
$$\approx \log_2(1+n/s) + s/2$$

Performance Analysis of Blocking Search

- If we use sequential search in both the index table and the blocks

$$ASL_b = \frac{b+1}{2} \qquad ASL_w = \frac{s+1}{2}$$

$$ASL = \frac{b+1}{2} + \frac{s+1}{2} = \frac{b+s}{2} + 1$$

$$= \frac{n+s^2}{2s} + 1$$

- When s = $\sqrt{n}$, we obtain the minimum ASL:

$$ASL = \sqrt{n} + 1 \approx \sqrt{n}$$

Performance Analysis of Blocking Search

- When n=10,000,
  - Sequential search takes 5,000 comparisons
  - Binary search takes 14 comparisons
  - Block search takes 100 comparisons

Pros & Cons of Blocking Search

- Pros:
  - Easy to insert and delete
  - Few movement of records
- Cons:
  - Space of a auxiliary array is needed
  - The blocks need to be sorted at the beginning
  - When a large number of insertion/deletion are done, or nodes are distributed unevenly, the efficiency will decrease.

Thinking

- Try comparing the sequential search witch binary search in terms of advantages and disadvantages.

- What are the application scenes of these search methods respectively?

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Set

- Set: a collection of well defined and distinct objects
- Search in a set: confirm whether a specific element belongs to the set

10.2 Search in a Set

	Names	Math Symbols	Computer symbols
Arithmetic operations	union	∪	+, |, *OR*
	intersection	∩	*, &, *AND*
	complement	−	−
	equality	=	==
	inequality	≠	!=
Logic operations	subset	⊆	<=
	superset	⊇	>=
	proper subset	⊂	<
	proper superset	⊃	>
	element of	∈	IN, at

Abstract Data Type of Sets

```
template<size_t N>                      // N is the number of elements of the set
class mySet {
public:
        mySet() ;                       // constructor
        mySet(ulong X);
        mySet<N>& set();                // set attributes of the set
        mySet<N>& set(size_t P, bool X = true);
        mySet<N>& reset();              // clear the set
        mySet<N>& reset(size_t P);      // delete the element p
        bool at(size_t P) const;        // belong operation
        size_t count() const;           // get the count of elements of the set
        bool none() const;              // check whether the set is empty
```

Abstract Data Type of Sets

```
bool operator==(const mySet<N>& R) const;           // equal
bool operator!=(const mySet<N>& R) const;           // not equal
bool operator<=(const mySet<N>& R) const;           // be subset of
bool operator< (const mySet<N>& R) const;           // be proper subset of
bool operator>=(const mySet<N>& R) const;           // be superset of
bool operator>(const mySet<N>& R) const;            // be proper superset of

friend mySet<N> operator&(const mySet<N>& L, const mySet<N>& R);  // union
friend mySet<N> operator|(const mySet<N>& L, const mySet<N>& R);  // intersection
friend mySet<N> operator-(const mySet<N>& L, const mySet<N>& R);  // complement
friend mySet<N> operator^(const mySet<N>& L, const mySet<N>& R);  // xor
};
```

Search in a Set

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	1	0	1	0	1	0	0	0	1	0	1	0	0

- Bitmap representation
  - Suitable when the number of valid elements is close to all the possible elements

Example: Find the Odd Primes between 0 and 15

Odd:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

&

Prime :

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	1	0	1	0	1	0	0	0	1	0	1	0	0

=

Odd prime :

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	1	0	1	0	1	0	0	0	1	0	1	0	0

Example: Represent a set by an UnsignedIinteger

- The complete set is a set with 40 elements
- The set {35, 9, 7, 5, 3, 1} can be represented with 2 ulongs.

0000 0000 0000 0000 0000 0000 0000 1000

0000 0000 0000 0000 0000 0010 1010 1010

Since 40 < 64, the size of 2 ulongs, we pad 0's on the left

10.2 Search in a Set

```
typedef unsigned long ulong;
enum {
    // number of bits of a unsigned long
    NB = 8 * sizeof (ulong),
    // The subscript of the last element of the
array
    LI = N == 0 ? 0 : (N - 1) / NB
};
    // the array used for saving the bit vector
ulong A[LI + 1];
```

Set the Elements of the Set

```
template<size_t N>
mySet<N>& mySet<N>::set(size_t P, bool X) {
        if (X)          // If X is true , the corresponding value of the bit vector should be set to 1
                A[P / NB] |= (ulong)1 << (P % NB);
                // a Union operation is operated for the element that  corresponds to p
        else    A[P / NB] &= ~((ulong)1 << (P % NB));
                //If X is false , the corresponding value of the bit vector should be set to 0
        return (*this);
}
```

Intersection Operations of a set "&"

```
template<size_t N>
mySet<N>& mySet<N>::operator&=(const mySet<N>& R)
{  // assignment of intersection
    for (int i = LI; i >= 0; i--)       // from low bits to high bits
        A[i] &= R.A[i];                 // intersect bit by bit in the unit
of ulongs
    return (*this);
}
template<size_t N>
mySet<N> operator&(const mySet<N>& L, const mySet<N>& R)
{   //intersection
    return (mySet<N>(L) &= R);
}
```

Thinking

- What else can we use to implement a set?
- Survey various implementations of set in the STL library.

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Search in a Hash Table

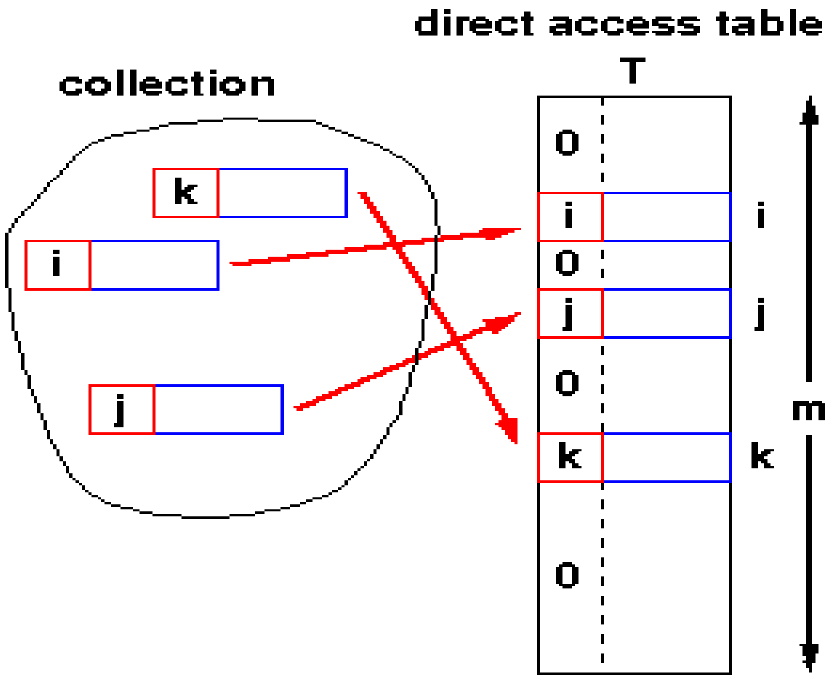
- 10.3.0 Basic problems in hash tables
- 10.3.1 Collision resolution
- 10.3.2 Open hashing
- 10.3.3 Closed hashing
- 10.3.4 Implementation of closed hashing
- 10.3.5 Efficiency analysis of hash methods

Basic problems in Hash Tables

- Search based on comparison of keys
  - Sequential search: ==, !=
  - Binary search, tree based: >, == , <
- Search is the operation interfaced with users
- When the problem size is large, the time efficiency of search methods mentioned above may become intolerable for users
- In the best case
  - Find the storage address of the record according to the key
  - No need to compare the key with candidate records one by one.

Think of Hash from Direct Access

- For example, we can get the element in an array with a specific subscript
  - Inspired by this, computer scientists invented hash method.
- A certain function relation h()
  - Keys of nodes k are used as independent variables
  - Function value h(K) is used as the storage address of the node
- Search uses this function to calculate the storage address
  - Generally, a hash table is stored in a one-dimensional array
  - The hash address is the array index



Example 1

Example 10.1: you already know the set of the key of a linear list: S = {and, array, begin, do, else, end, for, go, if, repeat, then, until, while, with}

We can let the hash table be: char HT2[26][8];

The value of hash function H(key), is the sequence number of the first letter of key in the alphabet {a, b, c, …, z}, which means H(key) = key[0] – ‘a’

Example 1 (continued)

Hash address	key
0	(and, array)
1	begin
2	
3	do
4	(end, else)
5	for
6	go
7	
8	if
9	
10	
11	

Hash address	key
13	
14	
15	
16	
17	repeat
18	
19	then
20	until
21	
22	(while, with)
23	
24	

Example 2

```
// the value of hash function is the average of the sequence numbers of
the first and the last letters of key in the alphabet. Which means:
int H3(char key[])
{
    int i = 0;
    while ((i<8) && (key[i]!='\0'))  i++;
    return((key[0] + key(i-1) - 2*'a') /2 )
}
```

Example 2 (continued)

Hash address	key
0	
1	and
2	
3	end
4	else
5	
6	if
7	begin
8	do
9	
10	go
11	for

Hash address	key
13	while
14	with
15	until
16	then
17	
18	repeat
19	
20	
21	
22	
23	
24	

Several Important Concepts

- The load factor $\alpha = N/\mathrm{M}$
  - M is the size of the hash table
  - N is the number of the elements in the table
- Collision
  - Some hash function return the same value for 2 or more distinct keys
  - In practical application, there are hardly any hash functions without collision
- Synonym
  - The two keys that collides with each other

Hash Function

- Hash function: the function mapping keys to storage addresses, generally denoted by h
- $Address = Hash(\ key\ )$
- Principles to select hash functions
  - Be easy to compute
  - The range of the function must be inside the range of the hash table
  - Try to map two distinct keys to different addresses as good as possible.

Various Factors Needed to be Consider

- Lengths of keys
- Size of hash tables
- Distribution of keys
- Frequency rate of searching for records
- …

Commonly-Used Hash Functions

- 1. Division method
- 2. Multiplication method
- 3. Middle square method
- 4. Digit analysis method
- 5. Radix conversion method
- 6. Folding method
- 7. ELF hash function

1. Division method

- Division method: divide M by key x, and take the remainder as the hash address, the hash function is:

$$h(x) = x \bmod M$$

- Usually choose a prime as M
  - The value of function relies on all the bits of independent variable x, not only right-most k bits.
  - Increase the probability of evenly distribution
  - For example, 4093

Why isn't M an even integer?

- If set M as an even integer?
  - If x is an even integer, h(x) is even too.
  - If x is an odd integer, h(x) is odd too;
- Disadvantages: unevenly distribution
  - If even integers occur more often than odd integers, the function values would not be evenly distributed
  - Vice versa

M shouldn't be a Power of Integers

$x \bmod 2^8$ choose right-most 8 bits

0110010111000011010

- If set M as a power of 2
  - Then, $h(x) = x \bmod 2^k$ is merely right-most k bits of x (represented in binary form)
- If set M to a power of 10
  - Then, $h(x) = x \bmod 10^k$ is merely right-most k bits of x (represented in decimal)
- Disadvantages: hashed values don't rely on the total bits of x

Problems of Division Method

- The potential disadvantages of division method
  - Map contiguous keys to contiguous values
- Although ensure no collision between contiguous keys
- Also means they must occupy contiguous cells
- May decrease the performance of hash table

2. Multiplication method

- Firstly multiply *key* by a constant A (0 < A < 1), extract the fraction part
- Then multiply it by an integer n, then round it down, and take it as the hash address
- The hash function is:
  - $hash(key) = \lfloor n * (A * key \% 1) \rfloor$
  - *"A * key % 1" denotes extracting the fraction part of A * key*
  - $A * key \% 1 = A * key - \lfloor A * key \rfloor$

Example

- let key = 123456, n = 10000 and let A = $(\sqrt{5}-1)/2$ = 0.6180339 ,
- Therefore,

$$hash(123456) =$$
$$= \lfloor 10000*(0.6180339*123456 \% 1) \rfloor =$$
$$= \lfloor 10000 * (76300.0041151... \% 1) \rfloor =$$
$$= \lfloor 10000 * 0.0041151... \rfloor = 41$$

Consideration about the Parameter Chosen in Multiplication Method

- If the size of the address space is p-digit then choose $n = 2^p$
  - The hash address is exactly the left-most p bits of the computed value
  - A * key % 1 = A * key - ⌊ A * key ⌋
  - Advantages: not related to choose of n
- Knuth thinks: A can be any value, it's related to the features of data waited to be sort. Usually golden section is the best

3. Middle Square Method

- Can use middle square method this moment: firstly amplify the distinction by squaring keys, then choose several bits or their combination as hash addresses.
- For example
  - A group of binary key: (00000100 , 00000110 , 000001010 , 000001001 , 000000111)
  - Result of squaring: (00010000 , 00100100 , 01100010 , 01010001 , 00110001)
  - If the size of the table is 4-digit binary number, we can choose the middle 4 bits as hash addresses: (0100, 1001, 1000, 0100, 1100)

4. Digit Analysis Method

- If there are *n* numbers, each with *d* digits and each digit can be one of *r* different symbols
- The occurring probabilities of these *r* symbols may are different
  - Distribution on some digits may be the same for the probabilities of all the symbols
  - Uneven on some digits, only some symbols occur frequently.
- Based on the size of the hash table, pick evenly distributed digits to form a hash address

Digit Analysis Method (2/4)

- The evenness of distribution of each digit λ_k

$$\lambda_k = \sum_{i=1}^{r} (\alpha_i^k - n/r)^2$$

  - α_i^k denotes the occurring number of ith symbols
  - n/r denotes expected value of all the symbols occurring on n digits evenly
- The smaller λ_k get, the more even the distribution of symbols on this digit is

Digit Analysis Method (3/4)

- If the range of hash table address is 3 digits, then pick the ④ ⑤ ⑥ digits of each key to form the hash address of the record
- We can add ① , ② , ③ digits to ⑤ digit, get rid of the carry digit, to become a 1-digit number. Then combine it with ④ , ⑥ digits, to form a hash address. Some other methods also work

9	9	2	1	4	8
9	9	1	2	6	9
9	9	0	5	2	7
9	9	1	6	3	0
9	9	1	8	0	5
9	9	1	5	5	8
9	9	2	0	4	7
9	9	0	0	0	1
①	②	③	④	⑤	⑥

①digit, $\lambda_1 = 57.60$
②digit, $\lambda_2 = 57.60$
③digit, $\lambda_3 = 17.60$
④digit, $\lambda_4 = 5.60$
⑤digit, $\lambda_5 = 5.60$
⑥digit, $\lambda_6 = 5.60$

Digit Analysis Method (4/4)

- Digit analysis method is only applied to the situation that you know the distribution of digits on each key previously
  - It totally relies on the set of keys
- If the set of keys changes, we need to choose again

5. Radix Conversion Method

- Regard keys as numbers using another radix.
- Then convert it to the number using the original radix
- Pick some digits of it as a hash address
- Usually choose a bigger radix as converted radix, and ensure that they are inter-prime.

Example: Radix Conversion Method

- For instance, give you a key $(210485)_{10}$ in base-10 system, treat it as a number in base-13 system, then convert it back into base-10 system
- $(210485)_{13}$

$$= 2\times13^5 + 1\times13^4 + 4\times13^2 + 8\times13 + 5$$

$$= (771932)_{10}$$

- If the length of hash table is 10000, we can pick the lowest 4 digits 1932 as a hash address

6. Folding Method

- The computation becomes slow if we use the middle square method on a long number
- Folding method
  - Divide the key into several parts with same length (except the last part)
  - Then sum up these parts (drop the carries) to get the hash address
- Two method of folding:
  - Shift folding — add up the last digit of all the parts with alignment
  - Boundary folding — each part doesn't break off, fold to and fro along the boundary of parts, then add up these with alignment, the result is a hash address

Example: Folding Method

- [example 10.6] If the number of a book is 04-42-20586-4

(a)	(b)
5 8 6 4	0 4 4 2 2 0 5 8 6 4
4 2 2 0	0 2 2 4 4 0
+ 0 4	+ 0 4
[1] 0 0 8 8	6 0 9 2
h(key)=0088	h(key)=6092

- (a) shift holding (b) Boundary holding

7. ELF hash function

- Used in the UNIX System V4.0 “Executable and Linking Format(ELF for short)
- int ELFhash(char* key) {

```
    unsigned long h = 0;
    while(*key) {
        h = (h << 4) + *key++;
        unsigned long g = h & 0xF0000000L;
        if (g) h ^= g >> 24;
        h &= ~g;
    }
    return h % M;
}
```

Features of ELF hash function

- Work well for both long strings and short strings
- Chars of a string have the same effect
- The distribution of positions in the hash table is even.

Application of Hash Functions

- Choose appropriate hash functions according to features of keys in practical applications
- Someone have used statistical analysis method of “roulette” to analyze them by simulation, and it turns out that the middle square is closest to “random”
  - If the key is not a integer but a string, we can convert it to a integer, then apply the middle square method

Thinking

- Consider when using hash methods:

  (1) how to construct (choose) hash functions to make nodes distributed evenly

  (2) Once collision occurs, how to solve it?

- The organization methods of the hash table itself

Chapter 10. Search

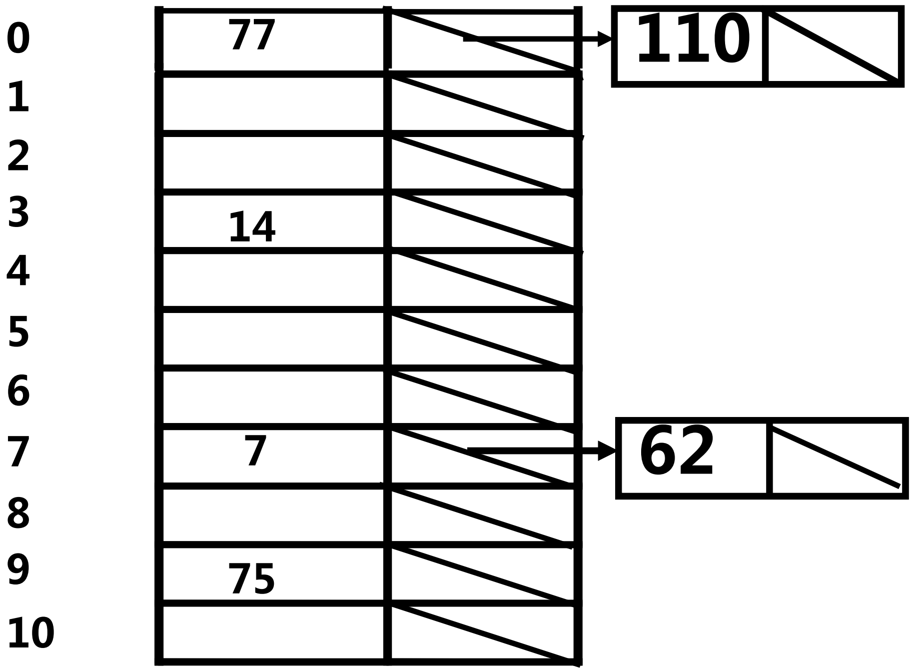
Search in a Hash Table

- 10.3.0 Basic problems in hash tables
- 10.3.1 Collisions resolution
- 10.3.2 Open hashing
- 10.3.3 Closed hashing
- 10.3.4 Implementation of closed hashing
- 10.3.5 Efficiency analysis of hash methods

Open Hashing

{77、14、75、 7、110、62 、95 }

■h(Key) = Key % 11



- The empty cells in the table should be marked by special values
  - like -1 or INFINITY
  - Or make the contents of hash table to be pointers, and the contents of empty cells are null pointers

Performance Analysis of Chaining Method

- Give you a table of size M which contains n records. The hash function (in the best case) put records evenly into the M positions of the table which makes each chain contains n/M records on the average
  - When M>n, the average cost of hash method is Θ(1)

10.3.3 Closed Hashing

- d_0=h(K) is called the base address of K.
- When a collision occurs, use some method to generate a sequence of hash addresses for key K

  d_1, d_2, ... d_i , ..., d_{m-1}

  - All the d_i (0<i<m) are the successive hash addresses
- With different way of probing, we get different ways to resolve collisions.
- Insertion and search function both assume that the probing sequence for each key has at least one empty cell
  - Otherwise it may get into a endless loop
- We can also limit the length of probing sequence

Problem may Arise - Clustering

- Clustering
  - Nodes with different hash addresses compete for the same successive hash address
  - Small clustering may merge into large clustering
  - Which leads to a very long probing sequence

Several General Closed Hashing Methods

- 1. Linear probing
- 2. Quadratic probing
- 3. Pseudo-random probing
- 4. Double hashing

1. Linear probing

- Basic idea:
  - If the base address of a record is occupied, check the next address until an empty cell is found
    - Probe the following cells in turn: d+1, d+2, ……, M-1, 0, 1, ……, d-1
  - A simple function used for the linear probing:
    $p(K,i) = I$
- Advantages:
  - All the cell of the table can be candidate cells for the new record inserted

Instance of Hash Table

- M = 15, h(key) = key%13
- In the ideal case, all the empty cells in the table should have a chance to accept the record to be inserted
  - The probability of the next record to be inserted at the 11th cell is 2/15
  - The probability to be inserted at the 7th cell is 11/15

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
26	25	41	15	68	44	6				36		38	12	51

Enhanced Linear Probing

- Every time skip constant c cells rather than 1
  - The ith cell of probing sequence is
    $(\mathrm{h}(K) + ic) \bmod M$
  - Records with adjacent base address would not get the same probing sequence
- Probing function is $\mathrm{p}(K,i) = i*c$
  - *Constant c and M must be co-prime*

Example: Enhance Linear Probing

- For instance, c = 2, The keys to be inserted, k_1 and k_2. $h(k_1) = 3$, $h(k_2) = 5$
- Probing sequences
  - The probing sequence of k_1: 3, 5, 7, 9, ...
  - The probing sequence of k_2: 5, 7, 9, ...
- The probing sequences of k_1 and k_2 are still intertwine with each other, which leads to clustering.

2. Quadratic probing

- Probing increment sequence: 1^2, -1^2, 2^2, -2^2, ..., The address formula is

  $d_{2i-1} = (d + i^2) \% M$

  $d_{2i} = (d - i^2) \% M$

- A function for simple linear probing :

  $p(K, 2i-1) = i*i$

  $p(K, 2i) = - i*i$

Example: Quadratic Probing

- Example: use a table of size M = 13
    - Assume for k_1 and k_2, $h(k_1)=3$, $h(k_2)=2$
- Probing sequences
    - The probing sequence of k_1: 3, 4, 2, 7, ...
    - The probing sequence of k_2: 2, 3, 1, 6, ...
- Although k_2 would take the base address of k_1 as the second address to probe, but their probing sequence will separate from each other just after then

3. Pseudo-Random Probing

- Probing function p(K,i) = perm[i - 1]
  - here perm is an array of length M – 1
  - It contains a random permutation of numbers between 1 and M

```
// generate a pseudo-random permutation of n numbers
void permute(int *array, int n) {
        for (int i = 1;  i <= n;  i ++)
                swap(array[i-1], array[Random(i)]);
}
```

Example: Pseudo-Random Probing

- Example: consider a table of size M = 13, perm[0] = 2, perm[1] = 3, perm[2] = 7.
  - Assume 2 keys k_1 and k_2, $h(k_1)=4$, $h(k_2)=2$
- Probing sequences
  - The probing sequence of k_1: 4, 6, 7, 11, ...
  - The probing sequence of k_2: 2, 4, 5, 9, ...
- Although k_2 would take the base address of k_1 as the second address to probe, but their probing sequence will separate from each other just after then

Secondary Clustering

- Eliminate the primary clustering
  - Probing sequences of keys with different base address overlap
  - Pseudo-random probing and quadratic probing can eliminate it
- Secondary clustering
  - The clustering is caused by two keys which are hashed to one base address, and have the same probing sequence
  - Because the probing sequence is merely a function that depends on the base address but not the original key.
  - Example: pseudo-random probing and quadratic probing

4. Double Probing

- Avoid secondary clustering
  - The probing sequence is a function that depends on the original key
  - Not only depends on the base address
- Double probing
  - Use the second hash function as a constant
    - $p(K, i) = i * h_2 (key)$
  - Probing sequence function
    - $d = h_1(key)$
    - $d_i = (d + i\ h_2 (key))\ \%\ M$

Basic ideas of Double Probing

- The double probing uses two hash functions h_1 and h_2
- If collision occurs at address $h_1(key) = d$, then compute $h_2(key)$, the probing sequence we get is :

  $(d+h_2(key)) \% M$, $(d+2h_2 (key)) \%M$, $(d+3h_2 (key)) \% M$, ...
- It would be better if h_2 (key) and M are co-prime
  - Makes synonyms that cause collision distributed evenly in the table
  - Or it may cause circulation computation of addresses of synonyms
- Advantages: hard to produce "clustering"
- Disadvantages: more computation

Method of choosing M and h2(k)

- Method1: choose a prime M, the return values of h_2 is in the range of

  $1 \le h2(K) \le M - 1$
- Method2: set $M = 2^m$,let h_2 returns an odd number between 1 and 2^m
- Method3: If M is a prime, $h_1(K) = K \bmod M$
  - $h_2(K) = K \bmod (M-2) + 1$
  - or $h_2(K) = [K / M] \bmod (M-2) + 1$
- Method4: If M is a arbitrary integer, $h_1(K) = K \bmod p$ (p is the maximum prime smaller than M)
  - $h_2(K) = K \bmod q + 1$ (q is the maximum prime smaller than p)

Thinking

- When inserting synonyms, how to organize synonyms chain?
- What kind of relationship do the function of double hashing h_2 (key) and h_1 (key) have?

Chapter 10. Search

- 10.1 Search in a list
- 10.2 Search in a set
- 10.3 Search in a hash table
- Summary

Search in a Hash Table

- 10.3.0 Basic problems in hash tables
- 10.3.1 Collision resolution
- 10.3.2 open hashing
- 10.3.3 closed hashing
- 10.3.4 Implementation of closed hashing
- 10.3.5 Efficiency analysis of hash methods

Implementation of Closed Hashing

Dictionary

- A special set consisting of elements which are two-tuples (key, value)
  - The keys should be different from each other (in a dictionary)
- Major operations are insertions and searches according to keys
  - **bool hashInsert(const Elem&);**

    // insert(key, value)
  - **bool hashSearch(const Key& , Elem&) const;**

    // lookup(key)

ADT of Hash Dictionaries (attributes)

```
template <class Key , class Elem , class KEComp , class
EEComp> class hashdict
{
private:
    Elem* HT;                    // hash table
    int M;                       // size of hash table
    int currcnt;                 // current count of elements
    Elem EMPTY;                  // empty cell
    int h(int x) const ;         // hash function
    int h(char* x)const ;        // hash function for strings
    int p(Key K , int i)         // probing function
```

ADT of Hash Dictionaries (methods)

```
public:
hashdict(int sz , Elem e) {                   // constructor
    M=sz;  EMPTY=e;
    currcnt=0;  HT=new Elem[sz];
    for (int i=0; i<M; i++) HT[i]=EMPTY;
}
~hashdict() { delete [] HT; }
bool hashSearch(const Key& , Elem&) const;
bool hashInsert(const Elem&);
Elem hashDelete(const Key& K);
int size() { return currcnt; }                // count of elements
};
```

Insertion Algorithm

hash function h, assume k is the given value

- If this address hasn't been occupied in the table, insert the record waiting for insertion into this address
- If the value of this address is equal to K, report "hash table already have this record"
- Otherwise, you can probe the next address of probing sequence according to how to handle collision, and keep doing this.
  - Until some cell is empty (can be inserted into)
  - Or find the same key (no need of insertion)

Code of Hash Table Insertion

```
// insert the element e into hash table HT
template <class Key, class Elem, class KEComp, class EEComp>
bool hashdict<Key, Elem, KEComp, EEComp>::hashInsert(const Elem& e) {
    int home= h(getkey(e));                        // home save the base address
    int i=0;
    int pos = home;                                // Start position of the probing sequence
    while (!EEComp::eq(EMPTY, HT[pos])) {
        if (EEComp::eq(e, HT[pos])) return false;
        i++;
        pos = (home+p(getkey(e), i)) % M;          // probe
    }
    HT[pos] = e;                                   // insert the element e
     return true;
}
```

Search Algorithm

- Similar to the process of insertion
  - Use the same probing sequence
- Let the hash function be h, assume the given value is K
  - If the space corresponding to this address is not occupied, then search fails
  - If not, compare the value of this address with K, if they are equal, then search succeeds
  - Otherwise, probe the next address of the probing sequence according to how to handle collision, and keep doing this.
    - Find the equal key, search succeeds
    - Haven't found when arrive at the end of probing sequence, then search fails

10.3 Search in a Hash Table

```
template <class Key, class Elem, class KEComp, class EEComp>
bool hashdict<Key, Elem, KEComp, EEComp>::
hashSearch(const Key& K, Elem& e) const {
    int i=0, pos= home= h(K);                    // initial position
    while (!EEComp::eq(EMPTY, HT[pos])) {
        if (KEComp::eq(K, HT[pos])) {            // have found
            e = HT[pos];
            return true;
        }
        i++;
        pos = (home + p(K, i)) % M;
    } // while
    return false;
}
```

Deletion

- Something to consider when delete records:
  - (1) The deletion of a record mustn't affect the search later
  - (2) The storage space released could be used for the future insertion
- Only open hashing (separated synonyms lists) methods can actually delete records
- Closed hashing methods can only make marks (tombstones), can't delete records actually
  - The probing sequence would break off if records are deleted. Search algorithm "until an empty cell is found (search fails)"
  - Marking tombstones increases the average search length

Problems Caused by Deletions

0	1	2	3	4	5	6	7	8	9	10	11	12
	K1	K2	K1		K2	K2	K2			K2		

- For example, a hash table of length M = 13, let keys be $k1$ and $k2$, h($k1$) = 2 , h($k2$) = 6。
- Quadratic probing
  - The quadratic probing sequence of $k1$: 2、3、1、6、11、11、6、5、12、...
  - The quadratic probing sequence of $k2$: 6、7、5、10、2、2、10、9、3、...
- Delete the record at the position 6, put the element in the last position 2 of $k2$ sequence instead, set position 2 to empty
- search $k1$, but fails (may be put at position 3 or 1 in fact)

Tombstones

- Set a special mark bit to record the cell status of the hash table
  - Be occupied
  - Empty
  - Has been deleted
- The mark to record the status of has been deleted is called tombstone
  - Which means it was occupied by some record ever
  - But it isn’t occupied now

Deletion Algorithms with Tombstones

```
template <class Key, class Elem, class KEComp, class EEComp>Elem
hashdict<Key,Elem,KEComp,EEComp>::hashDelete(const Key& K)
{   int i=0,  pos = home= h(K);                    // initial position
    while (!EEComp::eq(EMPTY, HT[pos]))  {
        if (KEComp::eq(K, HT[pos])){
            temp = HT[pos];
            HT[pos] = TOMB;                        // set up tombstones
             return temp;                          // return the target
        }
        i++;
        pos = (home + p(K, i)) % M;
    }
    return EMPTY;
}
```

Insertion Operation with Tombstones

- If a cell marked as a tombstone is met at the time of insertion, can we insert the new record into this cell?
  - In order to avoid inserting two same keys
  - The process of search should carry on along the probing sequence, until find a real empty cell

An Improved Version of Insertion Operation with Tombstones

```
template <class Key, class Elem, class KEComp, class EEComp>
bool hashdict<Key, Elem, KEComp, EEComp>::hashInsert(const
Elem &e)  {
    int insplace, i = 0, pos = home = h(getkey(e));
    bool tomb_pos = false;
    while (!EEComp::eq(EMPTY, HT[pos])) {
        if (EEComp::eq(e, HT[pos])) return false;
        if (EEComp::eq(TOMB, HT[pos]) &&  !tomb_pos)
        {insplace = pos;  tomb_pos = true;}            // The first
        pos = (home + p(getkey(e), ++ i)) % M;
    }
    if (!tomb_pos) insplace=pos;                       // no tombstone
       HT[insplace] = e;    return true;
}
```

Efficiency Analysis of Hash Methods

- Evaluation standard: the number of record visits needed for insertion, deletion, search
- Insertion and deletion operation of hash tables are both based on search
  - Deletion: must find the record at first
  - Insertion: must find until t the tail of the probing sequences, which means need a failed search for the record
    - For the situation without consideration about deletion, it is the tail cell.
    - For the situation with consideration about deletion, also need to arrive at the tail to confirm whether there are repetitive records

Important Factors Affecting Performance of Search

- Expected cost of hash methods is related to the load factor
- $\alpha= N/M$
  - When α is small, the hash table is pretty empty, it's easy for records to be inserted into empty base addresses.
  - When α is big, inserting records may need collision resolution strategies to find other appropriate cells
- With the increase of α, more and more records may be put further away from their base addresses

Analysis of Hash Table Algorithms (1)

- The probability of base addresses being occupied is α
- The probability of the i-th collision occurring is

$$\frac{N(N-1)\cdots(N-i+1)}{M(M-1)\cdots(M-i+1)}$$

- If N and M are both very large, then it can be expressed approximately as

$$(N/M)^i$$

- The expected value of the number of probing is 1, plus occurring probability of each the i-th ($i \geq 1$) collision, which is cost of inserting, :

$$1+\sum_{i=1}^{\infty}(N/M)^i = 1/(1-a)$$

Analysis of Hash Table Algorithms (2)

- A cost of successful search (or deletion) is the same as the cost of insertion
- With the increase of the number of records of hash tables, α also get larger and larger
- We can get the average cost of insertion (the average of the cost of all the insertion) by computing the integral from 0 to current value of α

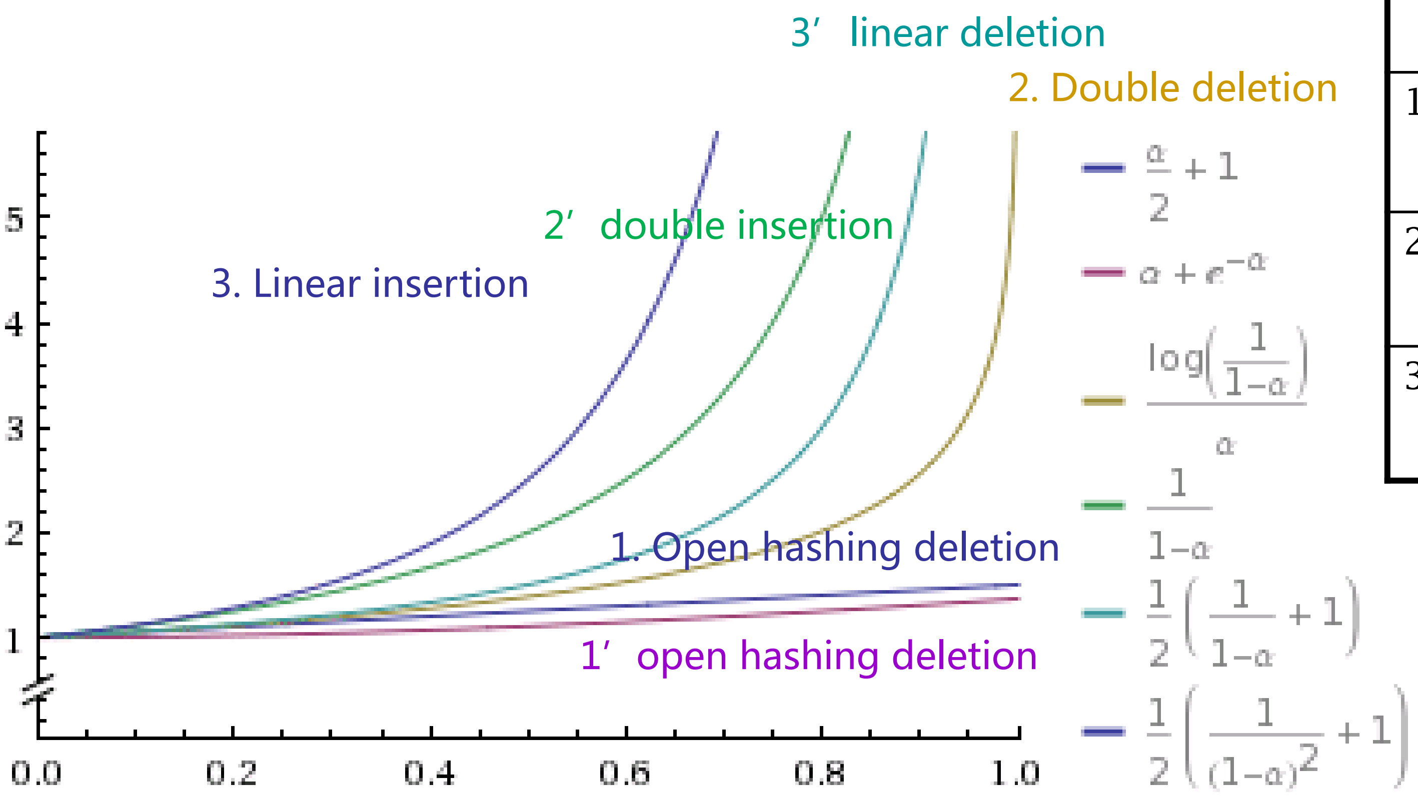
$$\frac{1}{a}\int_0^a \frac{1}{1-x}dx = \frac{1}{a}\ln\frac{1}{1-a}$$

Hash Table Algorithms Analysis (table)

No.	Collision resolution strategy	Successful search (deletion)	Failed search (insertion)
1	Open hashing	$1+\frac{\alpha}{2}$	$\alpha+e^{-\alpha}$
2	Double hashing	$\frac{1}{\alpha}\ln\frac{1}{1-\alpha}$	$\frac{1}{1-\alpha}$
3	Linear probing	$\frac{1}{2}\left(1+\frac{1}{1-\alpha}\right)$	$\frac{1}{2}\left(1+\frac{1}{(1-\alpha)^2}\right)$

Hash Table Algorithms Analysis (diagram)

- ASLs of using different way to resolve collision in hash tables



No.	Collision resolution strategy	Successful search (deletion)	Failed search (insertion)
1	Open hashing	$1+\frac{\alpha}{2}$	$\alpha+e^{-\alpha}$
2	Double hashing	$\frac{1}{\alpha}\ln\frac{1}{1-\alpha}$	$\frac{1}{1-\alpha}$
3	Linear probing	$\frac{1}{2}\left(1+\frac{1}{1-\alpha}\right)$	$\frac{1}{2}\left(1+\frac{1}{(1-\alpha)^2}\right)$

Conclusion of Hash Table Algorithms Analysis

- Normally the cost of hash methods is close to the time of visiting a record. It is very effective , greatly better than binary search which need log *n* times of record visit
  - Not depend on n, only depend on the load factor $\alpha=n/M$
  - With the increase of α, expected cost would increase too
  - When $\alpha \leq 0.5$, The excepted cost of most operations is less than 2 (someone say 1.5)
- The practical experience indicates that the critical value of the load factor α is 0.5 (close to half full)
  - When the load factor is bigger than this critical value, the performance would degrade rapidly

Conclusion of Hash Table Algorithms Analysis (2)

- If the insertion or deletion of hash tables is complicated, then efficiency degrades
  - A mass of insertion operation would make the load factor increases.
    - Which also increase the length of synonyms linked chains, and also increase ASL
  - A mass of deletion would increase the number of tombstones.
    - Which increase the average length from records to their base addresses
- In the practical application, for hash tables with frequent insertion or deletion, we can perform rehashing for hash tables regularly
  - Insert all the records to another new table
    - Clear tombstones
    - Put the record visited most frequently on its base address

Thinking

- Can we mark the status of empty cell and having been deleted as a special value, to distinguish them from "occupied" status?
- Survey implementation of dictionary other than hash tables.

Data Structures and Algorithms

Thanks

the National Elaborate Course (Only available for IPs in China)
http://www.jpk.pku.edu.cn/pkujpk/course/sjjg/
Ming Zhang, Tengjiao Wang and Haiyan Zhao
Higher Education Press, 2008.6 (awarded as the "Eleventh Five-Year" national planning textbook)