

# Unit 2: Consumer Theory

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## 1 Consumer demand

### 1.1 The experienced utility function

- Suppose that the consumer only cares about two goods:
  - $x$ , good of interest, measured in units
  - $m$ , a measure of all other consumption in \$. Think of it as amount of \$ spent on all other goods except for  $x$ .
  - Assume  $x \geq 0$ , but can have  $m \stackrel{\geq}{\leq} 0$ .
- Experienced utility function:  $U(x, m)$  = level of experienced utility (= satisfaction or happiness) from consumption bundle  $x, m$ .
- Assume  $U$  is quasi-linear:
  - $U(x, m) = B(x) + m$ ,  $B' > 0, B'' < 0$ .
- Remarks:
  - $B(x)$  = willingness to pay for  $x$  units of good  $x$
  - $B(0)$  need not be zero
  - $\frac{\partial U}{\partial m} = 1$ , i.e. constant marginal utility;  $\frac{\partial U}{\partial x} = B'(x)$ , i.e. decreasing marginal utility since  $B'' < 0$ .
  - $U(x, m)$  measured in dollars

## 1.2 Utility maximization problem

- Assume consumer is rational; i.e. chooses action to maximize experienced utility
- Consumer's problem, given wealth  $W$  and a price  $p$  per unit of good  $x$ :

$$\begin{aligned} & \max_{x \geq 0, m} U(x, W - xp) \\ & \sim \max_{x \geq 0} B(x) + W - xp && \text{by QL assumption} \\ & \sim \max_{x \geq 0} B(x) - xp && \text{by dropping constant } W \end{aligned}$$

- Familiar structure: benefit minus cost
- Notation:  $x^*(p)$  = optimal choice = demand for  $x$  at  $p$ .
- It follows that  $m^*(p) = W - px^*(p)$
- Solution:
  - Concavity conditions satisfied
  - Case 1: Interior ( $x^*(p) > 0$ ): when  $B'(0) > p$  and  $B'$  crosses  $p$ .
  - Case 2: Corner solution at  $x^*(p) = 0$ : when  $B'(0) < p$ .
  - Case 3:  $x^* = \infty$ : when  $B'(x) > p$  for all  $x$ .  
Not very interesting, since crossing conditions usually satisfied via  $B'(\infty) = 0$
  - Economic intuition: starting at zero, keep buying as long as  $MB > MC$

## 1.3 The demand function

Demand curve for individual consumer

- Demand function:  $x^*(p)$ , defined for all  $p > 0$
- In economics, we always graph price on the vertical axis, and quantity on the horizontal axis
- Key result: Demand function  $\sim$  MB curve

- With  $x$  as independent variable, graph  $B'$
- Taking  $p$  as independent variable, same locus of points gives the demand function.

## 1.4 Example

- $U(x, m) = A \ln x + m$
- Observe that all solutions must be interior, since  $B'(0) = \infty > p$ .
- Solving  $B'(x) = p$  yields  $x^*(p) = \frac{A}{p}$
- $A$  is a taste parameter. As  $A$  increases the demand curve shifts right (i.e., the amount demanded increases at any  $p$ ).

## 1.5 Example

- $U(x, m) = A \ln x + m$
- What is the demand function if there is a mandatory minimum purchase of at least 1 unit of  $x$  ?
- For  $p \leq A$ , demand curve the same as in previous example,  $x_{bef}^*(p) = x_{after}^*(p)$
- For  $p > A$ , demand curve now is  $x_{after}^*(p) = 1 > x_{bef}^*(p)$ . In this case, since the consumers are forced to buy more than they want, they are made worse off by the policy.

## 1.6 Properties of the demand function

- RESULT: Law of Demand:  $\frac{dx^*}{dp} \leq 0$ , w/ inequality if  $x^* > 0$ .  
Outline of proof:
  - For prices in the range of corner solutions:  $x^*(p) = 0 \implies \frac{dx^*}{dp} = 0$
  - $x^*(p) > 0 \implies$  interior solution  $\implies B'(x^*(p)) = p$   
Taking derivatives of both sides w.r.t.  $p$  yields  $B'' \frac{dx^*}{dp} = 1$   
 $\implies \frac{dx^*}{dp} = \frac{1}{B''}$   
Therefore  $B'' < 0 \implies \frac{dx^*}{dp} < 0$

- RESULT: No income effects:  $\frac{dx^*}{dW} = 0$ 
  - Why? With quasilinear preferences, the FOC that characterizes  $x^*$  does not depend on how much of the good  $m$  is consumed
- Both results approximately hold for goods which account for a small fraction of consumer's overall spending
- Both can fail when  $U$  is non-quasilinear

## 2 Consumer surplus

### 2.1 Consumer Surplus: Basics

- How do we measure well-being of consumer?
- Consumer surplus = NET benefit of buying optimally at price  $p$ , measured in dollars
- Mathematically, it is defined as follows:

$$\begin{aligned}
 CS(p) &= U(x^*(p), W - px^*(p)) - U(0, W) \\
 &= B(x^*(p)) - B(0) - px^*(p) && \text{by quasilinearity} \\
 &= \int_0^{x^*(p)} B'(x)dx - \int_0^{x^*(p)} p \, dx && \text{the Fundamental Theorem of Calculus} \\
 &= \int_0^{x^*(p)} (B'(x) - p) \, dx && \text{by linearity of the integral}
 \end{aligned}$$

- Definition in the first line says  $CS(p)$  equals experienced utility of buying  $x^*(p)$  minus experienced utility of buying zero.
- Graphical description: CS = area between  $B'$  and  $p$ , from  $x = 0$  to  $x = x^*$
- Properties of CS:
  - $CS \geq 0$
  - $CS > 0$  iff  $x^*(p) > 0$

- Can see both from the graphical description
- Can also see it from a *revealed preference* argument:  $(0, W)$  is always feasible, i.e. the consumer can always buy nothing. Therefore CS can't be negative: if the consumer buys anything, it must make her better off.

## 2.2 Example

- Suppose  $U(x, m) = 2A\sqrt{x} + m$ 
  - $B' = \frac{A}{\sqrt{x}}$
  - $x^*(p) = \frac{A^2}{p^2}$
- Consider two different ways of computing  $CS(p)$ .
- Direct method:

$$\begin{aligned}
 CS(p) &= U(x^*(p)) - U_{no\,trade} \\
 &= \left[ 2A\sqrt{x^*(p)} + W - px^*(p) \right] - [0 + W] \\
 &= \frac{2A^2}{p} - \frac{A^2}{p^2}p \\
 &= \frac{A^2}{p}
 \end{aligned}$$

- Graphical/integral method:

$$\begin{aligned}
 CS(p) &= \int_0^{x^*(p)} (B'(x) - p) dx \\
 &= \int_0^{\frac{A^2}{p^2}} \left( \frac{A}{\sqrt{x}} - p \right) dx \\
 &= \left[ 2A\sqrt{x} - px \right]_0^{\frac{A^2}{p^2}} \\
 &= \frac{A^2}{p}
 \end{aligned}$$

## 2.3 Effect of price changes on consumer surplus

- Consider effect on CS of price decrease from  $p_0$  to  $p_1$
- It is given by:

$$\begin{aligned}\Delta CS_{p_0 \rightarrow p_1} &= CS(p_1) - CS(p_0) \\ &= \text{Change in experienced utility following price change from } p_0 \text{ to } p_1 \\ &= x_0^* \Delta p + \int_{x_0^*}^{x_1^*} (B' - p_1) dx\end{aligned}$$

where  $\Delta p = p_0 - p_1$

- First term = value of buying the old  $x_0^*$  units at lower price  $p_1$
- Second term = value of buying additional units at price  $p_1$

## 2.4 More on consumer surplus

- Let's extend the notion of consumer surplus to more general situations
- Let  $\theta$  = complete description of consumer's problem
- $x^*(\theta)$  = optimal choice at situation  $\theta$
- Now,  $CS(\theta) = U(x^*(\theta)) - U_{no-trade}$
- Example:

–  $U(x, m) = 2A\sqrt{x} + m$

–  $\theta$  : price =  $p$ , mandatory minimum purchase = 1 unit

– Now, we have

$$x^*(\theta) = \begin{cases} 1 & \text{if } p \geq A \\ \frac{A^2}{p^2} & \text{if } p < A \end{cases}$$

and

$$CS(\theta) = \begin{cases} 2A - p & \text{if } p \geq A \\ \frac{A^2}{p} & \text{if } p < A \end{cases}$$

- Important lesson from example: There are situations in which CS can be negative. They often involve situations in which consumers are forced to make purchases that they would not make voluntarily.

### 3 Recovering preferences from data

#### 3.1 What if we don't know the utility function?

- In reality, we never know consumers' experienced utility functions. However, we can infer changes in consumer surplus from behavior alone!

#### 3.2 How to compute CS using only behavioral data

- Given observations of several price/quantity pairs, can estimate the function  $x^*(p)$  using statistics
- Then, construct the inverse demand function  $p^*(x) = x^{*-1}(p)$
- But we know that at an interior allocation,  $p^*(x) = B'(x)$ , as long as the consumer makes decisions by maximizing her experienced utility function
- It follows that

$$CS(p) = \int_0^{x^*(p)} (p^*(x) - p) dx$$

- This gives CS as a function of  $x^*(p), p^*(\cdot)$ , and  $p$ , all of which are observable
- Can use this to estimate the change in consumer surplus that would follow from an unobserved change in prices, the introduction of a new tax, etc.

#### 3.3 Recovering the utility function from observed behavior

- Suppose  $x^*(\cdot)$  observed.
- We recover  $U$  using the following steps:
  - Assume  $U$  is quasi-linear, i.e.  $U(x, m) = B(x) + m$
  - Then  $p^*(x) = B'(x)$
  - So  $B(x) = B(0) + \int_0^x p^*(u) du$

$$- U(x, m) = \int_0^x p^*(u) du + m + \text{constant}$$

- So we can recover  $U(x, m)$  up to a constant, which is equal to  $B(0)$ .
- Constant typically unimportant in most applications.
- What if  $U$  is not quasilinear?
  - OK if  $U \approx$  quasi-linear, even if not exactly so
  - Method generalizes (taught in advanced courses, requires substantially more math)

### 3.4 Example

- Consider an example of how to recover  $U(x, m)$  from  $x^*(p)$
- Suppose we observe that  $x^*(p) = \frac{10}{p} - 1$
- Then  $p^* = \frac{10}{x+1}$
- So  $B(x) = 10 \ln(x + 1)$ , and  $U(x, m) = 10 \ln(x + 1) + m + \text{constant}$

### 3.5 Example

- $\theta_0$  = initial situation at which consumer buys freely at price  $p$
- At initial situation observe  $x^*(p) = \frac{9}{p} - 1$ .
- $\theta_1$  = new situation at which individuals buy freely, but also get 2 free units (regardless of how much they buy)
- Question: Predict what is  $CS(\theta_1)$ ?
- Solve in three steps.
- Step 1: Recover  $U(x, m)$  from the initial observed demand.
  - Get  $U(x, m) = 9 \ln(x + 1) + m$  (similar to previous example)
- Step 2: Predict  $x_1^*(p)$  by maximizing the recovered preferences



- Important:  $x_1^*(p)$  denotes the amount bought, not the amount consumed which is equal to  $x_1^*(p) + 2$ .
- Get

$$x_1^*(p) = \begin{cases} 0 & \text{if } p \geq 3 \\ \frac{9}{p} - 3 & \text{if } p < 3 \end{cases}$$

- Step 3: Predict  $CS(\theta_1)$

- Get

$$CS(\theta_1) = \begin{cases} 9\ln 3 & \text{if } p \geq 3 \\ 9\ln \frac{9}{p} + 9 - 3p & \text{if } p < 3 \end{cases}$$

### 3.6 Application: Valuing new products

- Hypothetical example: compute the value of introducing household robots for a typical US consumer

- Data:

	$p$	$x$
2020	9000	1
2025	1000	9

- Graph the data, and fit a line through them (trivial in this example, since only two points)
- Get  $x^*(p) = \max\{0, 10 - \frac{p}{1000}\}$
- Value of introducing robots =  $CS(p) - CS(p = \infty) = CS(p)$  .
- Using graphical method:
  - $CS_{2020} = \frac{1}{2} \cdot 1 \cdot 1000 = \$500$
  - $CS_{2025} = \frac{1}{2} \cdot 9 \cdot 9000 = \$40,500$
- Lesson: Value of a new product depends strongly on the price at which it's sold.

## 4 Consumer mistakes

### 4.1 Decision mistakes

- A simple model of decision mistakes:
  1. Decision Utility vs. Experienced Utility
    - Experienced utility: describes well-being/hedonics
    - Decision utility: describes objective that is maximized at decision time
  2.  $x^*(\theta) = \max_x U^{DU}(x)$  s.t. feasibility constraints in  $\theta$   
 $x^{opt}(\theta) = \max_x U^{EU}(x)$  s.t. feasibility constraints in  $\theta$
  3. Rational behavior:  $U^{DU} = U^{EU} \implies x^* = x^{opt}$
  4. Mistakes:  $U^{DU} \neq U^{EU} \implies x^* \neq x^{opt}$
- Example:
  - $U^{EU} = 2A\sqrt{x} + m$
  - $U^{DU} = 4A\sqrt{x} + m$
  - So  $x^{opt}(p) = \frac{A^2}{p^2}$ ,  $x^*(p) = \frac{4A^2}{p^2}$ . Note the gap between the consumer choice and the optimal choice.
- Remarks:
  1.  $B^{DU}(x) \neq B^{EU}(x) + constant \implies$  mistakes arise
  2. If mistakes, then:
    - Utility function recovered from behavior  $\neq U^{EU}$
    - CS calculated with the recovered utility function is incorrect
  3. Thus, critical to know if mistakes likely in a given context
- Example (continued):
  - The CS estimated under the assumption that the consumer is rational is  $CS^{est}(p) = \frac{4A^2}{p}$ .
  - But the true CS, given the consumer's true EU function, is  $CS^{true}(p) = 0$ .

## 4.2 Application: Addiction

- Here is a simple model to think about the consumption of addictive substances:
  - $U^{EU}(x, m) = B(x) + m$
  - $U^{DU}(x, m) = \theta B(x) + m, \theta \gg 1$
  - Can have  $CS^{true}(p) < 0$ , in which case prohibition can be welfare improving!

## 5 Final remarks

- Key ideas:
  1. Modeling behavior:  $x^*(\theta)$  given by  $\max_{x \geq 0} U^{DU}(x)$  s.t. constraints in  $\theta$
  2. Measuring well-being: Consumer surplus,  $CS(\theta) = U^{EU}(x^*(\theta)) - U_{no-trade}^{EU}$
  3. Given rationality, can recover  $B(x)$  from  $x^*(p)$  and measure CS from  $p^*(x)$
- Tips on problem solving:
  - Must specify maximization problem correctly
  - Consumer behavior ( $x^*(\theta)$ ) follows from maximizing  $U^{DU}$  given the constraints in situation  $\theta$
  - Optimal level of consumption ( $x^{opt}(\theta)$ ) follows from maximizing  $U^{EU}$  given the constraints in situation  $\theta$
  - Don't mix up demand  $x^*(p)$  and inverse demand  $p^*(x)$
  - Careful when computing CS using the notion of 'area under the curve'