Quantum Mechanics & Quantum Computation

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Lecture 4: Bell Inequalities

EPR and Bell

Einstein, Podolsky, Rosen (EPR) Paradox (1935)

Local hidden variable theory

Bell 1965

There is an experiment that distinguishes between the predictions of quantum mechanics and any local hidden variable theory.

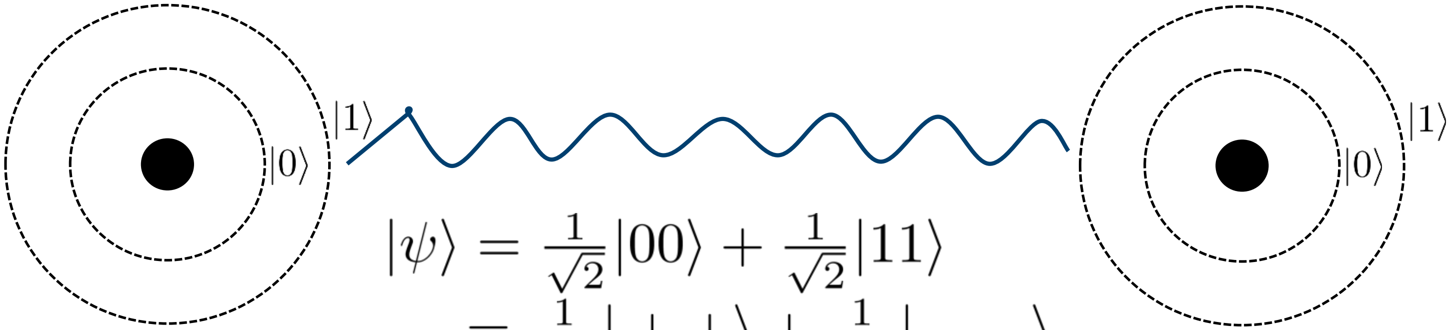
Clausser, Horne, Shimony, Holt 1969

Simplification

Aspect 1982

Experiment

No Signaling Theorem



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\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
&= \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle
\end{aligned}
$$

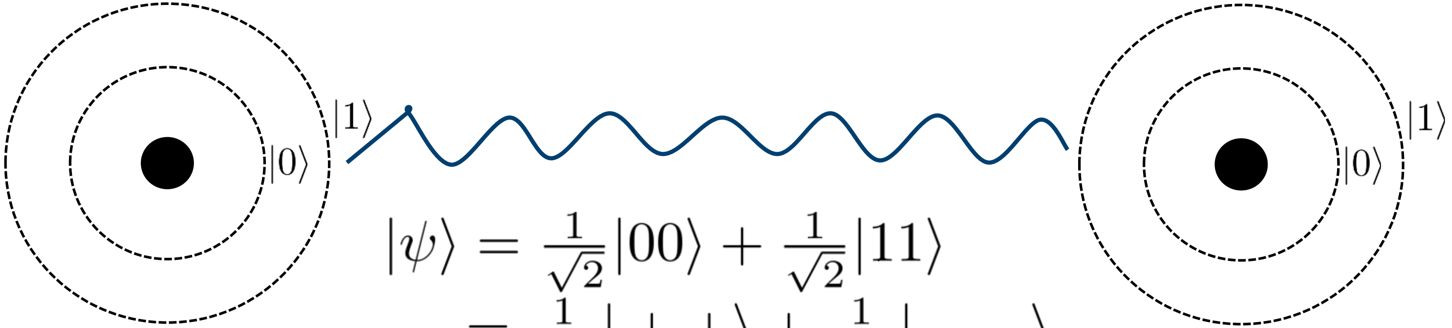
$P[0] = \frac{1}{2}$

$|0\rangle$

Cannot use entanglement to send a message faster than the speed of light.

But can use it to create non-classical correlations...!

Bell State



$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
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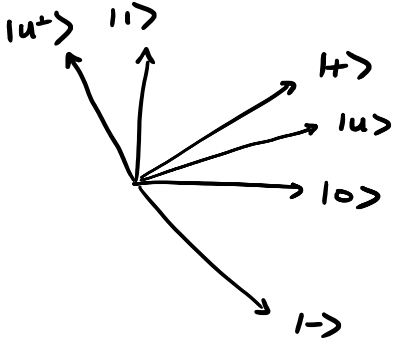
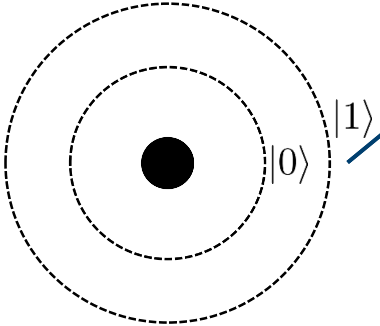
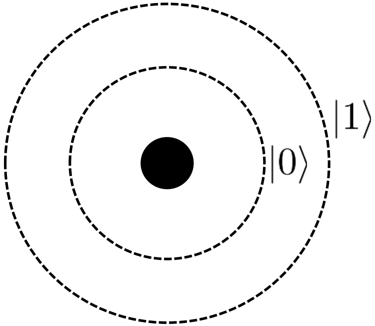


$|u\rangle = a_0|0\rangle + a_1|1\rangle \qquad a_0, a_1 \in \mathbb{R} \qquad a_0^2 + a_1^2 = 1.$

$$P[u] = ? \rightsquigarrow P[u] = ?$$

Rotational Invariance of Bell State



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\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
&= \underline{\frac{1}{\sqrt{2}}|uu\rangle + \frac{1}{\sqrt{2}}|u^{\perp}u^{\perp}\rangle}
\end{aligned}
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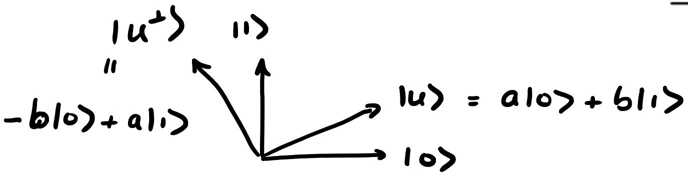


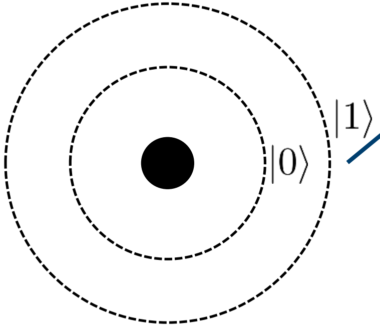
$a^2 + b^2 = 1$.

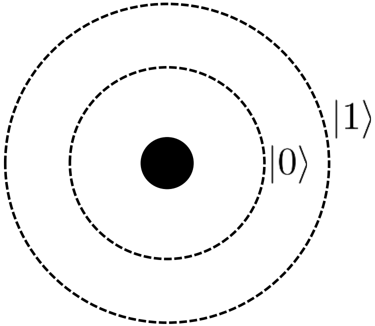
$$\frac{1}{\sqrt{2}}\Big((a|0\rangle + b|1\rangle)(a|0\rangle + b|1\rangle) + (-b|0\rangle + a|1\rangle)(-b|0\rangle + a|1\rangle)\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[a^2|00\rangle + ab\cancel{|01\rangle} + ba\cancel{|10\rangle} + b^2|11\rangle\Big) + \Big(b^2|00\rangle - \cancel{ba|01\rangle} - ab\cancel{|10\rangle} + a^2|11\rangle\Big)\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[(a^2+b^2)|00\rangle + (a^2+b^2)|11\rangle\Big]$$

Rotational Invariance of Bell State



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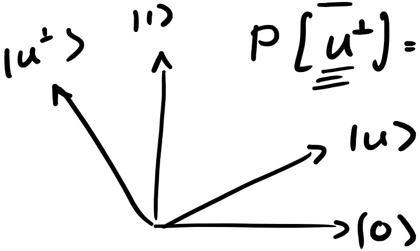
$P[\underline{\underline{u}}] = \frac{1}{2}$

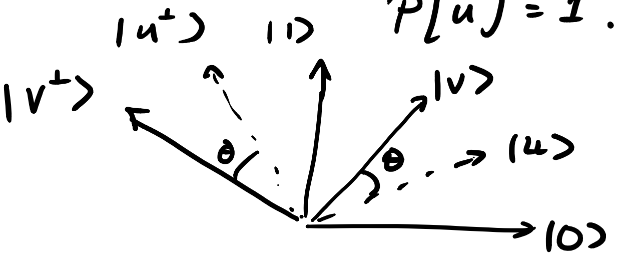
$P[\underline{\underline{u^{\perp}}}] = \frac{1}{2}$



New State $= |uu\rangle$

Measure 2nd qubit.

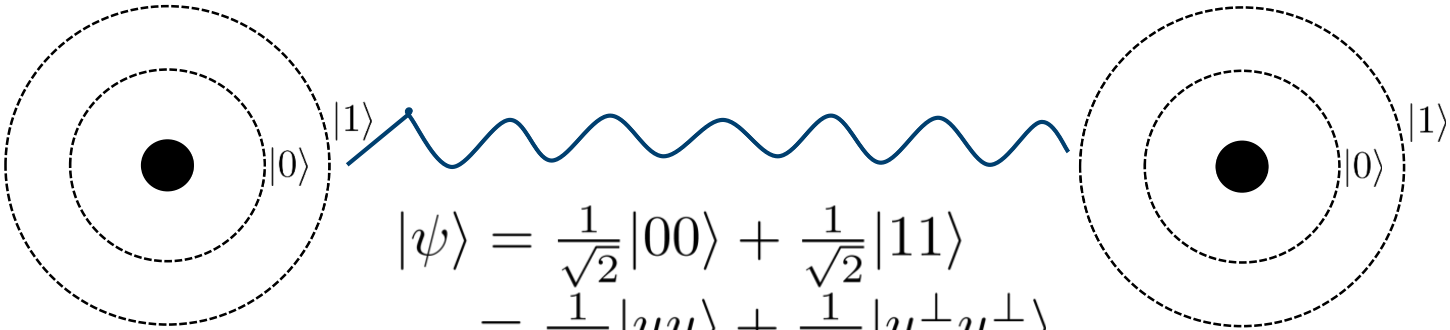
$P[u] = 1$.



u : $P[v] = \cos^2\theta$

$u^{\perp}$: $P[v^{\perp}] = \cos^2\theta$.

Rotational Invariance of Bell State



$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
$$= \frac{1}{\sqrt{2}}|uu\rangle + \frac{1}{\sqrt{2}}|u^{\perp}u^{\perp}\rangle$$

$|u\rangle = a|0\rangle + b|1\rangle \quad a, b \in \mathbb{R}$

✓ $|u\rangle = \alpha|0\rangle + \beta|1\rangle \quad \alpha, \beta \in \mathbb{C}$.

$\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$ — singlet state.

$|u\rangle, |u^{\perp}\rangle \qquad = \frac{1}{\sqrt{2}}|u u^{\perp}\rangle - \frac{1}{\sqrt{2}}|u^{\perp}u\rangle$.

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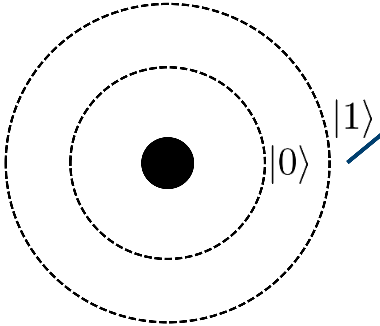
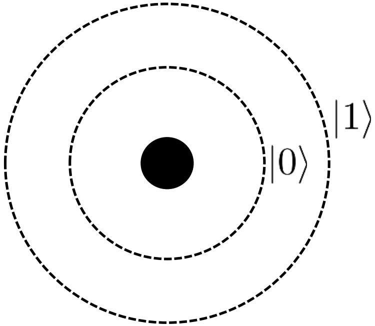
Rotational invariance of Bell state II

Rotational Invariance of Bell State





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$P[u] = \frac{1}{2}$

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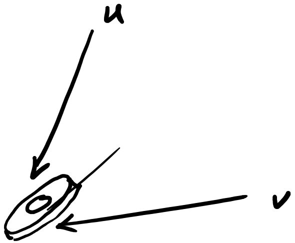
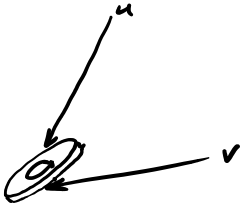
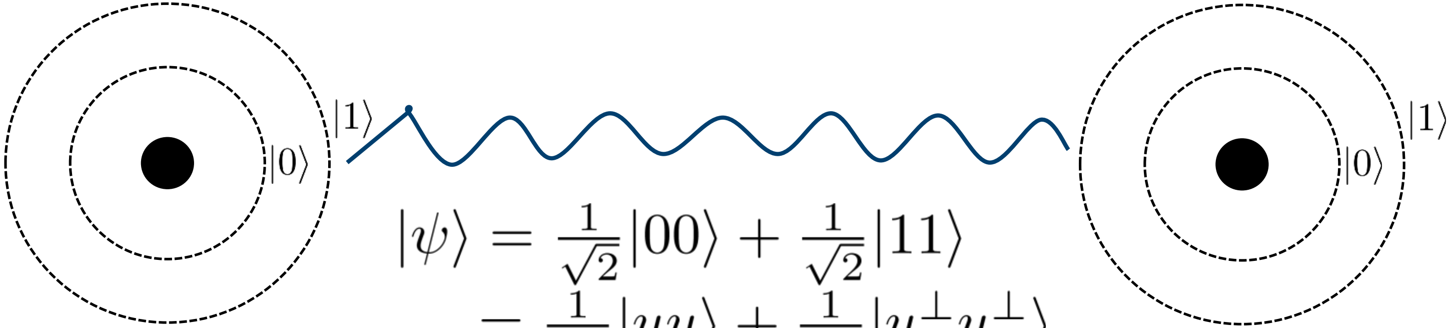




Rotational Invariance of Bell State



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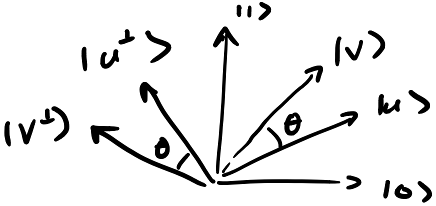
$|u\rangle, |u^{\perp}\rangle$ basis

$|v\rangle, |v^{\perp}\rangle$ basis.

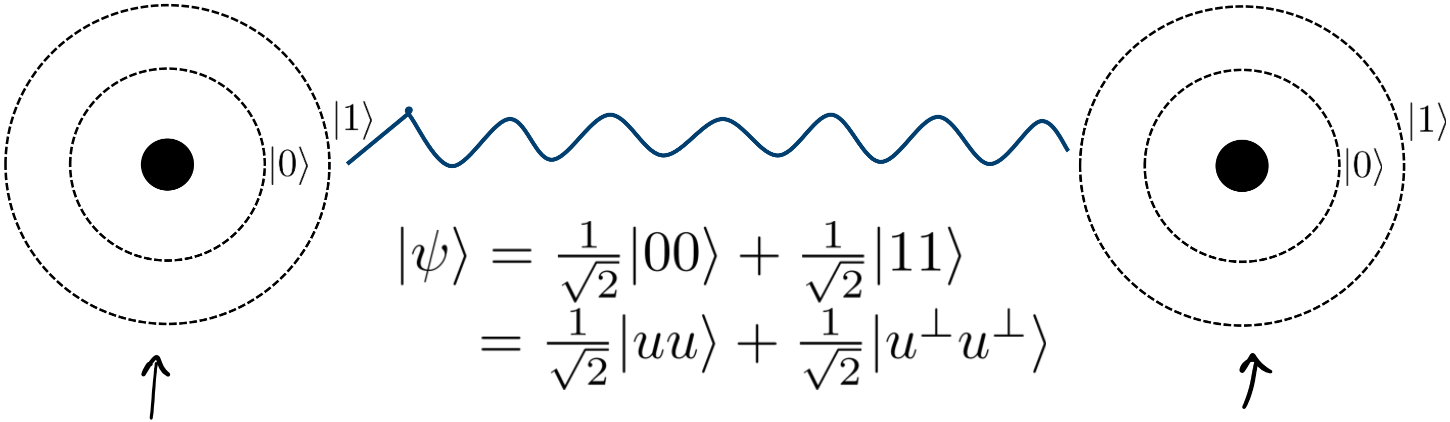


Prob. see corresponding outcomes $= \cos^2\theta$.

u, v

or $u^{\perp}, v^{\perp}$

No Signaling Theorem



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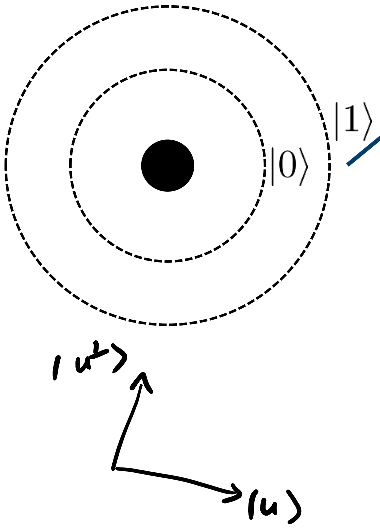
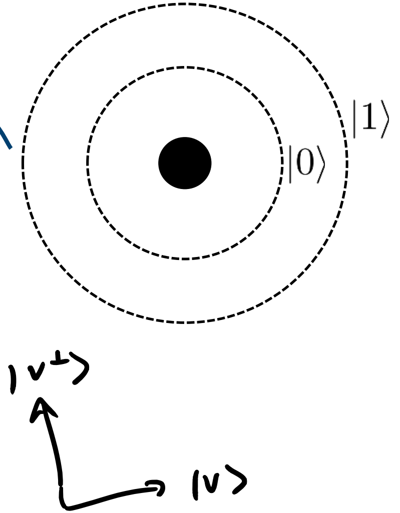
Cannot use entanglement to send a message faster than the speed of light.

But can use it to create non-classical correlations...!

Rotational Invariance of Bell State





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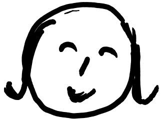
CHSH Inequality

CHSH Game



Inputs	$x \in \{0,1\}$	$y \in \{0,1\}$
Ouput	$a \in \{0,1\}$	$b \in \{0,1\}$

<u>Win</u> $x = y = 1$ and $a \neq b$

or $\overline{x \text{ or } y} = 0$ and $a = b$

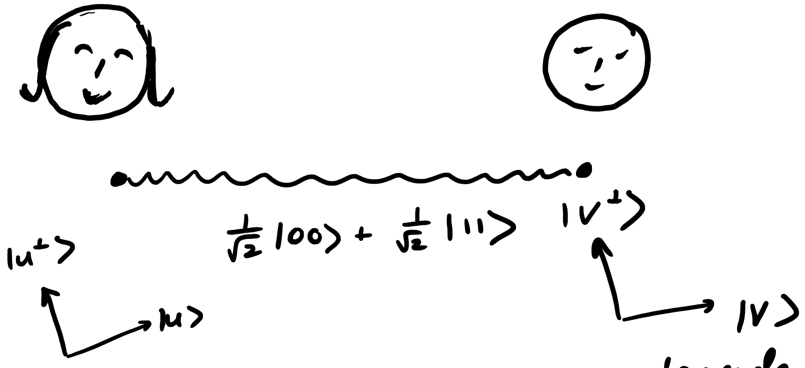
max $P[\, x \cdot y = a \oplus b \,]$

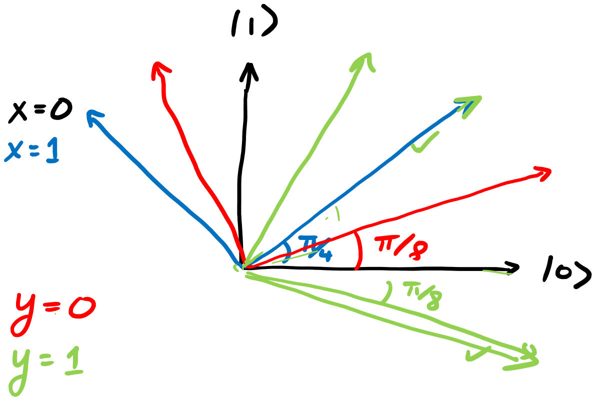
Classically: $\leq 3/4$.

Quantum $\cos^2 \pi/8$

$\approx$ .85.

CHSH Game





u depends upon input x.

depends upon y.

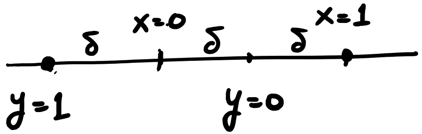
u & v chosen : if $x \cdot y = 0$ $\quad \theta = \pi/8 \longrightarrow$ same wp $\cos^2 \pi/8$.

$x = y = 1$ $\quad \theta = 3\pi/8 \longrightarrow$ same wp $\cos^2 \frac{3\pi}{8}$

$= \sin^2 \pi/8$

diff wp $1 - \sin^2 \pi/8$

$= \cos^2 \pi/8$.



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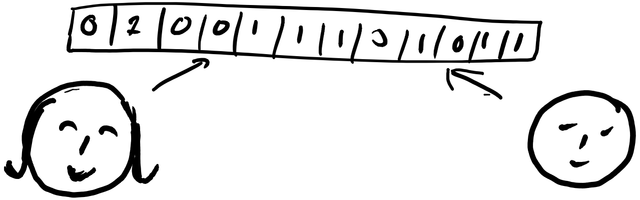
Bell & local realism

CHSH Game



x	y	a	b
0	0	0	0
✓ 0	1	0	0
1	0	0	0
1	1	0 ×	0

Inputs $\quad x \in \{0,1\} \qquad y \in \{0,1\}$

Ouput $\quad \underline{a} \in \{0,1\} \qquad \underline{b} \in \{0,1\}$

<u>Win</u> $\quad x = y = 1$ and $a \neq b$

or $\overline{x \lor y} = 0$ and $a = b$

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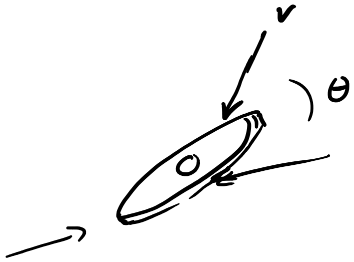
$\boxed{\text{Classically: } \le 3/4.}$

Quantum $\quad \cos^2 \pi/8$

$\approx$

$.85$ ✓



$$P[\text{diff outcomes}] = \frac{2\theta}{2\pi} = \frac{\theta}{\pi}.$$

Quantum $\quad P[\text{diff}] = \sin^2\theta \approx \theta^2 \quad \theta$ small.

$$\theta^2 << \frac{\theta}{\pi}$$