A Proof of that Linearizability is a Compositional Condition

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Compositionality

- A correctness condition CC is compositional if:
  - History $H$ satisfies CC iff every register subhistory $H|x$ satisfies CC

- Linearizability is compositional:
  - $\text{LIN}(H) \iff \forall x: \text{LIN}(H|x)$

- We will prove that this is the case
Definition of Linearizability

• Remember the definition of linearizability:
  – LIN(H) iff there exists a sequential history S satisfying the following requirements:
    (1) S is legal
    (2) S and H are equivalent
    (3) If \( o_1 <_H o_2 \) then \( o_1 <_S o_2 \)

  – \( o_1 <_H o_2 \) denotes that \( \text{res}(o_1) \rightarrow_H \text{inv}(o_2) \)
  • \( e_1 \rightarrow_H e_2 \) denotes that \( e_1 \) precedes \( e_2 \) in \( H \)
The Proof

• We must show that:
  – \( \text{LIN}(H) \Rightarrow \forall x: \text{LIN}(H|x) \), and
  – \( \text{LIN}(H) \iff \forall x: \text{LIN}(H|x) \)

• The \( \Rightarrow \) direction follows trivially
  – Exercise: Complete the proof
    • Assume \( \text{LIN}(H) \), implying that there exists a sequential history \( S \) satisfying requirements (1)-(3)
    • Show that subhistory \( S|x \) satisfies the requirements of a sequential history for \( \text{LIN}(H|x) \)
The $\Leftrightarrow$ Direction

• We must show that $\text{LIN}(H) \Leftrightarrow \forall x: \text{LIN}(H|x)$

• Assume that the right-hand side holds:
  – For each $x$, there exists a sequential history $S_x$ that satisfies the requirements of $\text{LIN}(H|x)$
    (1) $S_x$ is legal
    (2) $S_x$ and $H|x$ are equivalent
    (3) If $o_1 <_{H|x} o_2$ then $o_1 <_{S_x} o_2$

• We will construct a sequential history $S$ that satisfies the requirements of $\text{LIN}(H)$
Constructing Operation Graph

• Create a graph, whose vertices are operations in $H$, and edges are added as follows:
  – Add an edge from $o$ to $o'$ if $o <_{Sx} o'$
  – Add an edge from $o$ to $o'$ if $o <_{H} o'$

  – We refer to an edge as a $<_{Sx}$ edge or a $<_{H}$ edge (there can be zero, one, or two edges from $o$ to $o'$)
Constructing Sequential History $S$

• If the constructed graph is acyclic, then a topological sort can be performed
  – Creates a total order on operations, compatible with the partial ordering in the graph
  – History $S$ is created directly from this total ordering

• Sequential history $S$ created in this way satisfies the requirements for LIN(H) by construction:
  – $S$ is legal since the total ordering in $<_x$ is legal,
  – $S$ and $H$ are equivalent,
  – $o_1 <_H o_2$ implies $o_1 <_S o_2$
Acyclic

• We need to show that any graph constructed as described is acyclic
  – So that the graph can be topologically sorted

  – Proof by contradiction: assume a minimal cycle exists of a certain length, and reach contradiction
Cycle of Length n=2

• Cycles with two operations:
  
  – $o_1 <_H o_2 <_H o_1$
    
    • Not possible, as $<_H$ is a partial order
  
  – $o_1 <_{sx} o_2 <_{sx} o_1$
    
    • Not possible, as $<_{sx}$ is a total order
  
  – $o_1 <_H o_2 <_{sx} o_1$
    
    • As $o_1$ and $o_2$ are both ops on $x$, then $o_1 <_H o_2$ implies that $o_1 <_{sx} o_2$, which is contradicted in previous case
Cycle of Length n=3

• Cycles with three operations:
  – $O_1 <_H O_2 <_H O_3 <_H O_1$
    • Not minimal as $<_H$ is partial order
    • Similar contradiction if $<_x$ instead of $<_H$
  – $O_1 <_H O_2 <_H O_3 <_x O_1$
    • Not minimal as $<_H$ is partial order
  – In fact any cycle of length three must have two consecutive edges of same type ($<_H$ or $<_x$), and therefore cannot be minimal
Cycle of Length $n \geq 4$

- Consider a cycle of arbitrary length $n \geq 4$
- At some point in the cycle there is a section:
  - $o_1 <_H o_2 <_{Sx} o_3 <_H o_4$
  - We will show that this cycle is not minimal, as there must exist an edge $o_1 <_H o_4$
- Focus on edge $o_2 <_{Sx} o_3$, there are two cases:
  - Either $o_2 <_H o_3$, and hence $o_1 <_H o_4$ by transitivity of $<_H$
  - Or, not($o_3 <_H o_2$), this case is handled on next slide
Cycle of Length n≥4, case 2

- Second case, continued from previous slide:
  - not(o₃ <ₜ o₂) implies that not(res(o₃) →ₜ inv(o₂))
  - As →ₜ is a total order, we have inv(o₂) →ₜ res(o₃)
  - Together with o₁ <ₜ o₂, and o₃ <ₜ o₄, we have:
    - res(o₁) →ₜ inv(o₂) →ₜ res(o₃) →ₜ inv(o₄)
    - Implying that res(o₁) →ₜ inv(o₄) ⇒ o₁ <ₜ o₄

- Hence, the cycle containing o₁ <ₜ o₂ <ₕ o₃ <ₜ o₄ is not minimal
Summary

• Cycles of all lengths have been contradicted and the graph is therefore acyclic
• It can be topologically sorted into a sequential history $S$ that meets requirements of $\text{LIN}(H)$

• We have proven that:
  – Linearizability is compositional
    • $\text{LIN}(H) \iff \forall x: \text{LIN}(H \mid x)$