Quantum Mechanics & Quantum Computation

Umesh V. Vazirani
University of California, Berkeley

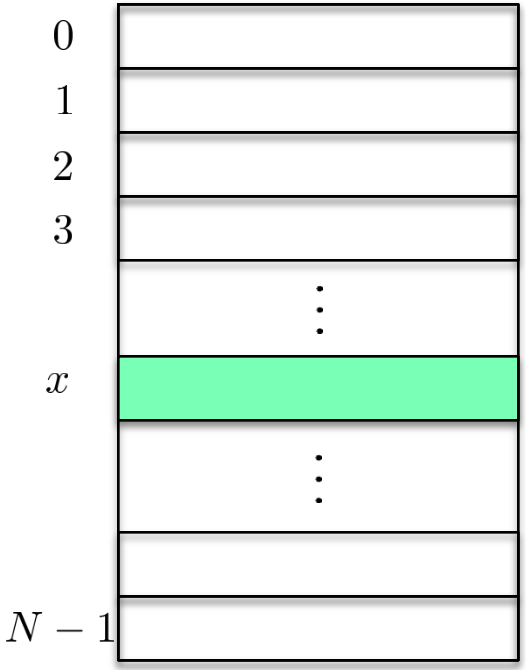
Lecture 11: Quantum Search

Needle in a haystack

Searching for a needle in a haystack

Unstructured search

"Digital haystack"



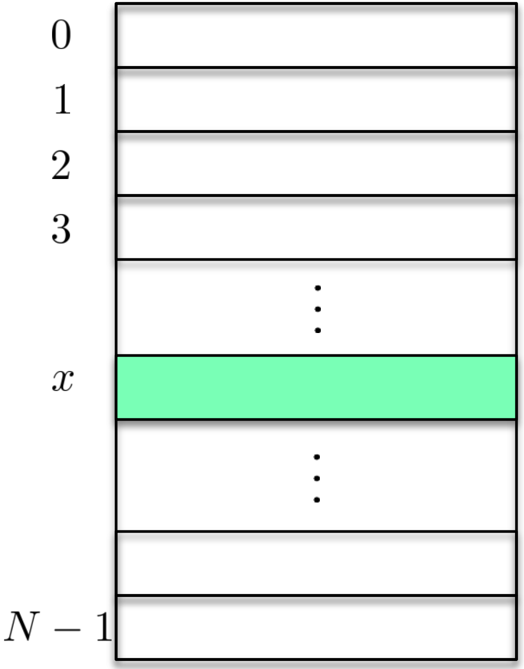
Goal: Search for the marked entry.

Classically: try random entries.
O(N/2) expected time.

Quantum??

Unstructured search

“Digital haystack”



Quantum??

Unstructured search

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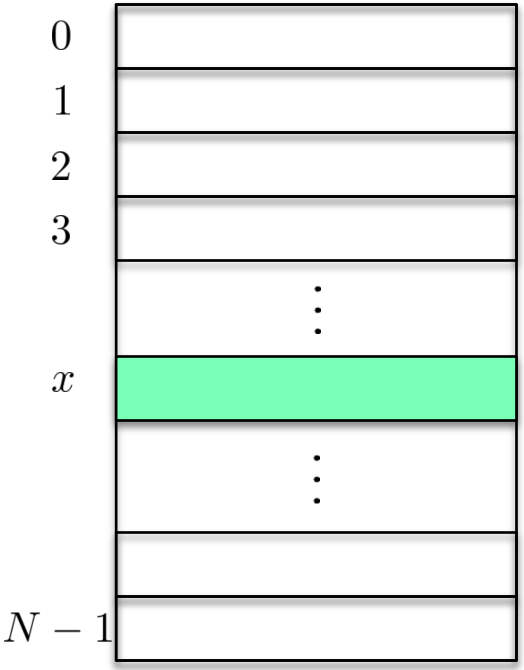


$N = 2^n$

NP-Complete Problems:

Satisfiability:

Finding a solution to an NP-complete problem can be viewed as a search problem.

$$f(x_1, \dots, x_n) = \underline{(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_5 \vee x_6) \wedge \cdots}$$

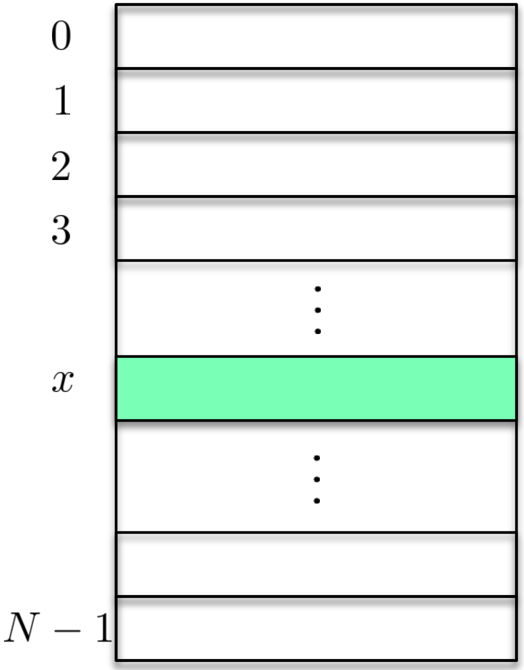
Is there a configuration of $x_1, x_2, \cdots$ that satisfy the above formula?

There are $\underline{\underline{2^n}}$ possible configurations.

$= N$

Unstructured search

"Digital haystack"



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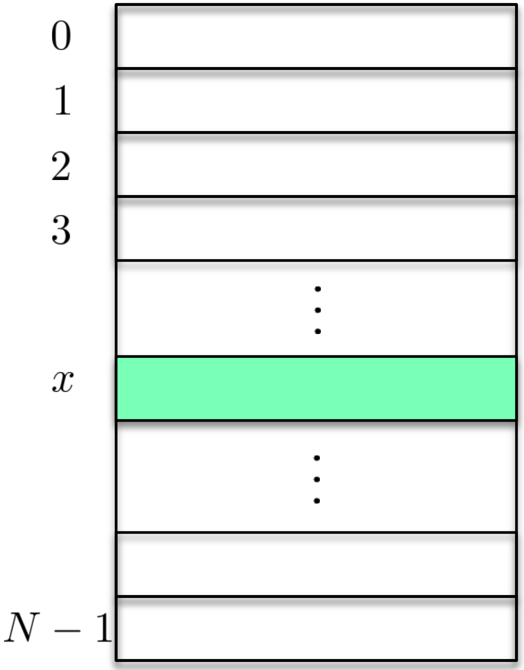


Quantum??

Grover's Algorithm: Quantum algorithm for unstructured search that takes $O(\sqrt{N})$ time.

$N = 2^n$

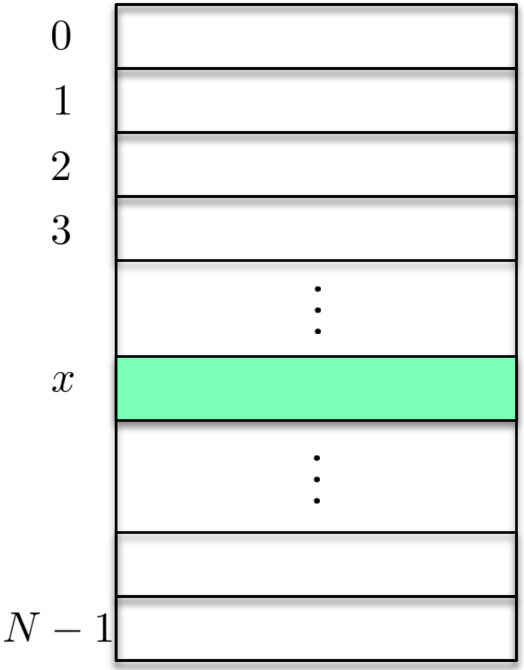
size n .

SAT

$\sqrt{N} = 2^{n/2}$

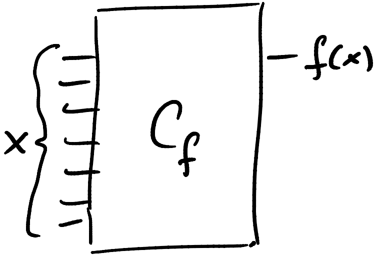
Unstructured search

"Digital haystack"

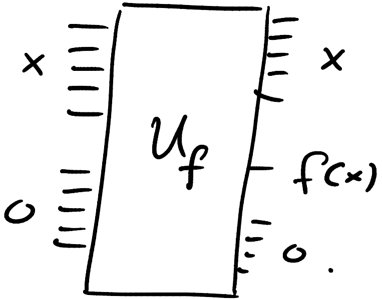


Problem. Given f:{0, 1, …, N-1} → {0,1}, find x: f(x) = 1.

Hardest case: There is exactly one x: f(x) = 1.



$N = 2^n$



$$\sum_x \alpha_x |x\rangle|0\rangle \longrightarrow \sum_x \alpha_x |x\rangle|f(x)\rangle$$

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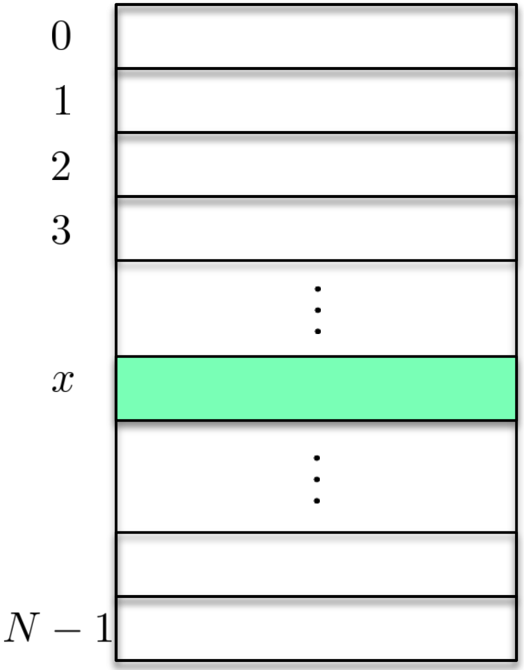


Problem. Given f:{0, 1, …, N-1} → {0,1}, find x: f(x) = 1.

Hardest case: There is exactly one x: f(x) = 1.

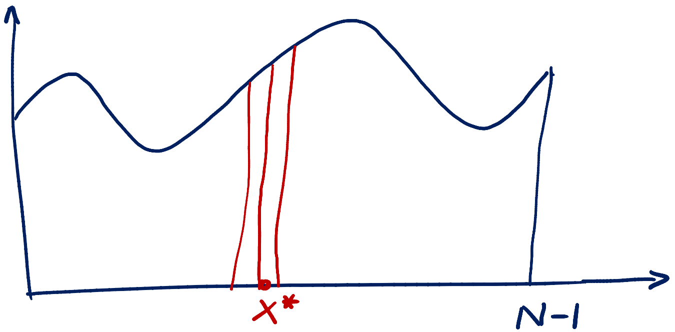
Phase Inversion

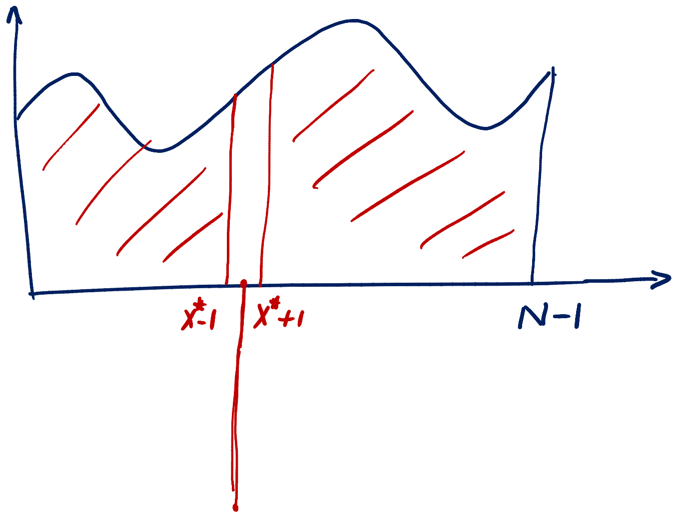
$$f(x^*) = 1$$

$$\sum_x \alpha_x |x\rangle$$

$\downarrow$ Phase inversion.

$$\sum_{x \neq x^*} \alpha_x |x\rangle \;-\; \alpha_{x^*} |x^*\rangle$$





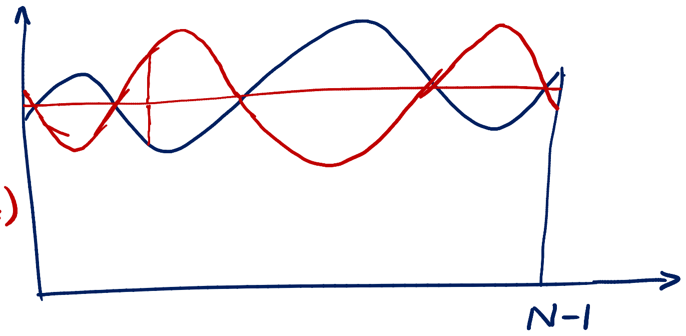
Inversion About Mean

$$\sum_x \alpha_x |x\rangle$$

$$\mu = \frac{\sum_{x=0}^{N-1} \alpha_x}{N}$$

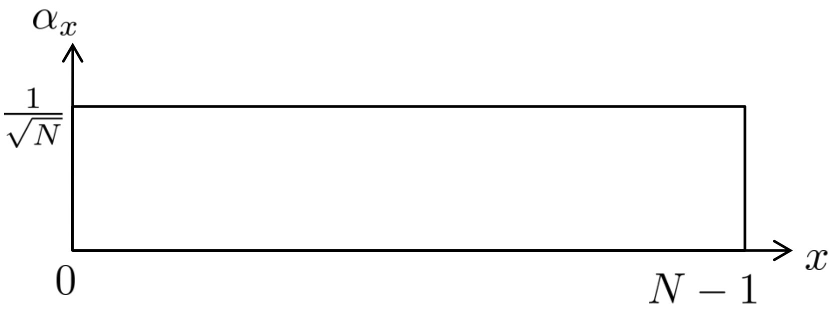
$$\alpha_x \longrightarrow (2\mu - \alpha_x) = \mu + (\mu - \alpha_x)$$

$$\sum_x \alpha_x |x\rangle \longrightarrow \sum_x (2\mu - \alpha_x) |x\rangle$$



Grover's algorithm

Problem. Given $f : \{0, \ldots, N-1\} \to \{0,1\}$ such that $f(x) = 1$ for exactly one x, find x.



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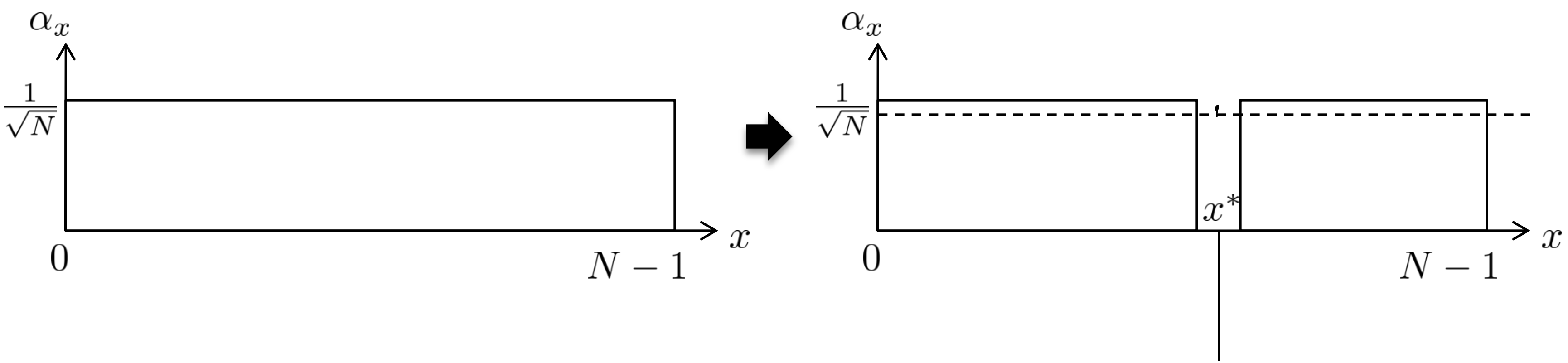

Grover's algorithm

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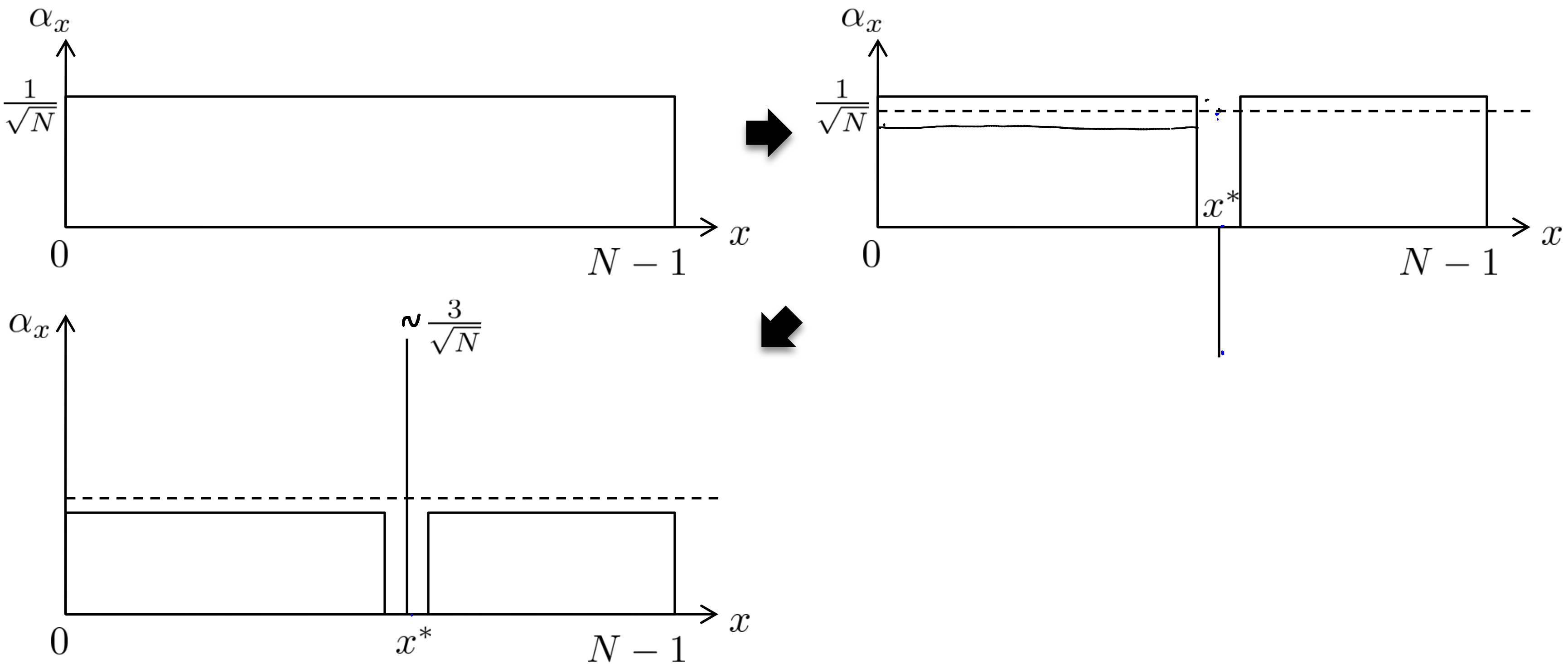

Grover's algorithm

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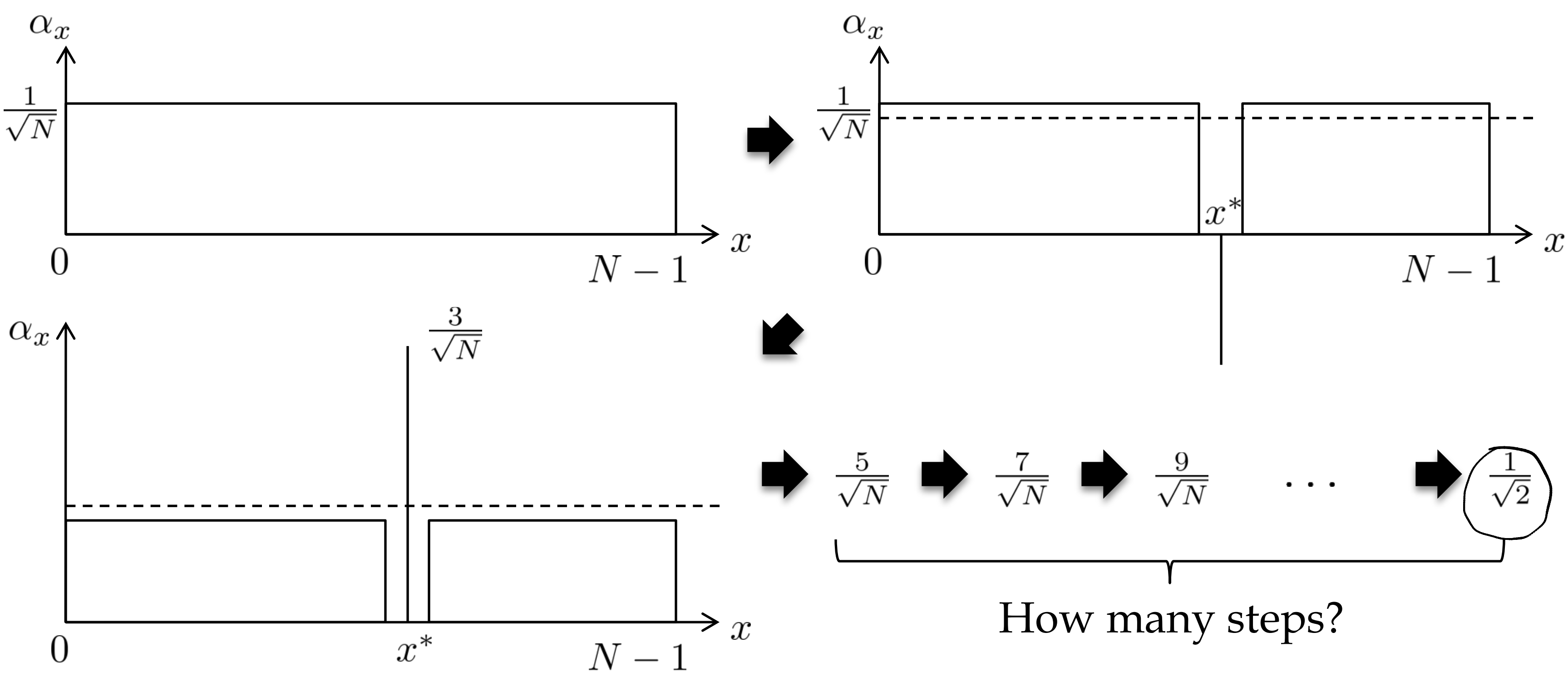
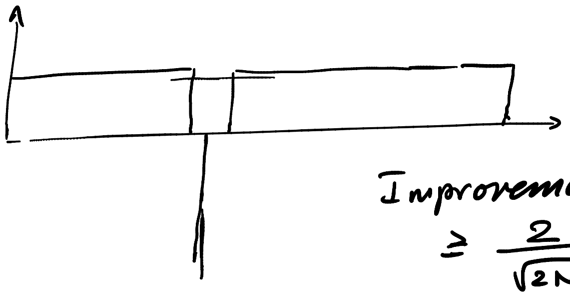


Grover's algorithm

What is the amplitude of the rest when the needle has $\frac{1}{\sqrt{2}}$? $\frac{1}{\sqrt{2N}}$

At this point how much improvement are we making per step?

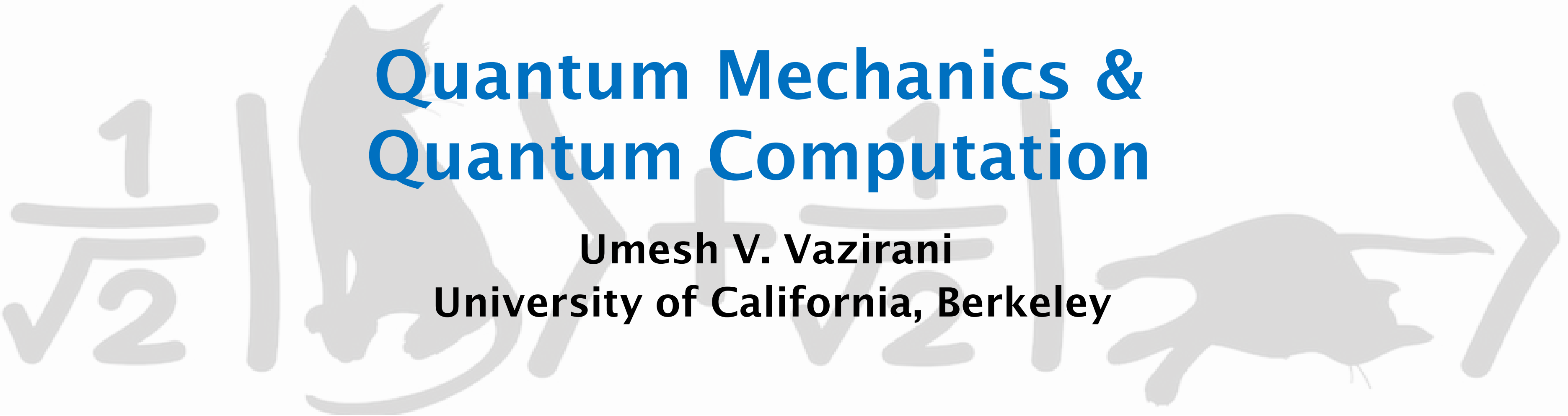


$\geq \frac{1}{\sqrt{2N}}$

Improvement/step $\geq \frac{2}{\sqrt{2N}} = \sqrt{\frac{2}{N}}$.

We will reach $\frac{1}{\sqrt{2}}$ in $O(\sqrt{N})$ steps.

$$\#\text{steps} \leq \frac{\frac{1}{\sqrt{2}}}{\sqrt{\frac{2}{N}}} = \frac{\sqrt{N}}{2}.$$



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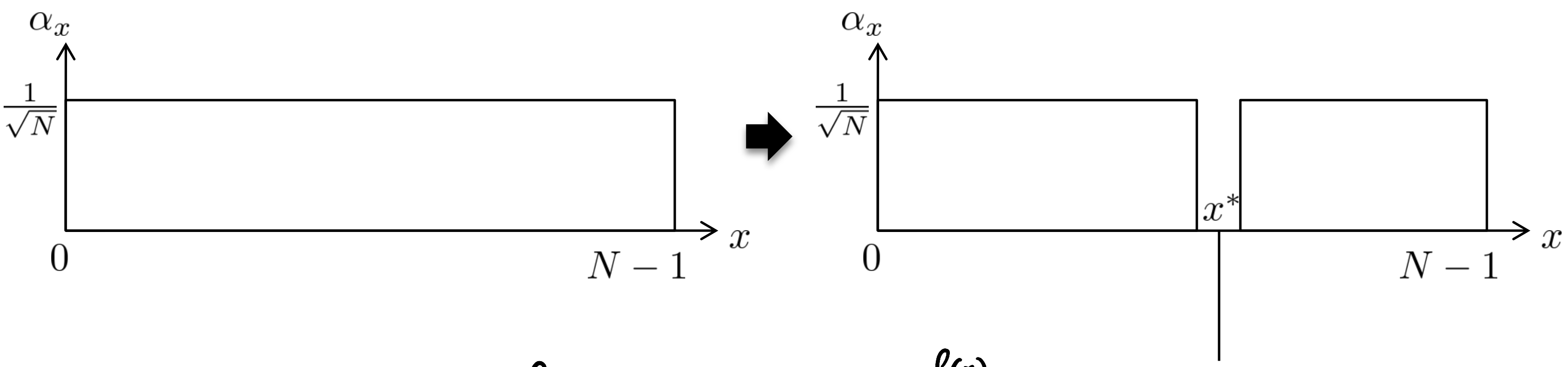
Implementing Grover's Algorithm

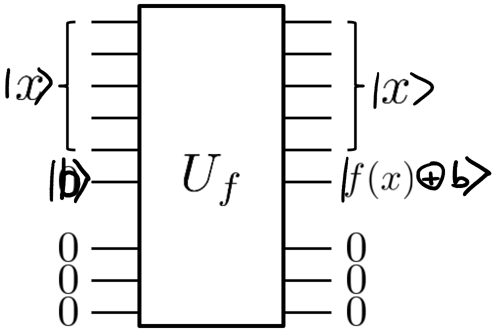
Phase Inversion

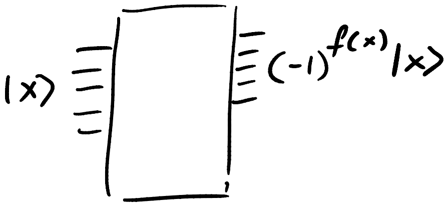
Problem. Given $f : \{0, \ldots, N-1\} \to \{0,1\}$ such that $f(x) = 1$ for exactly one x, find x.



$$\sum_x \alpha_x |x\rangle \xrightarrow[\text{inversion}]{\text{Phase}} \sum_x \alpha_x \underline{\underline{(-1)}}^{f(x)} |x\rangle$$

Phase Inversion





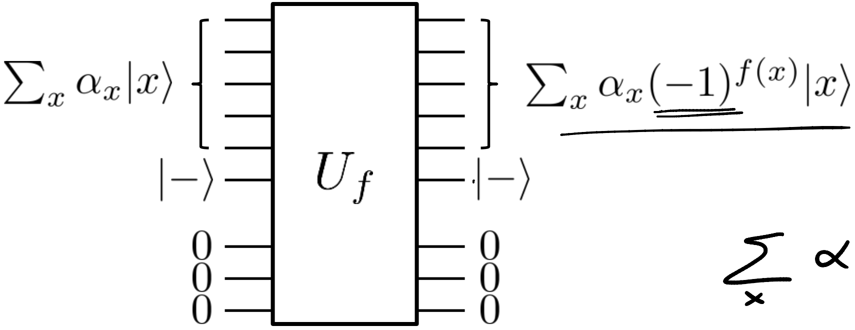
b \ f(x)	0	1
0	0	1
1	1	0
$\lvert -\rangle$	$\lvert -\rangle$	$-\lvert -\rangle$

$\left.\right\}$ output $= (-1)^{f(x)}|-\rangle$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\downarrow$$

$$\underbrace{\frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle} = -\underbrace{\left[\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right]}_{= |-\rangle}$$



$$\sum_x \alpha_x |x\rangle \underbrace{(-1)^{f(x)}} |-\rangle$$

Inversion About Mean

$$\sum_x \alpha_x |x\rangle$$

$$\mu = \frac{\sum_{x=0}^{N-1} \alpha_x}{N}$$

$$\alpha_x \longrightarrow (2\mu - \alpha_x) = \mu + (\mu - \alpha_x)$$

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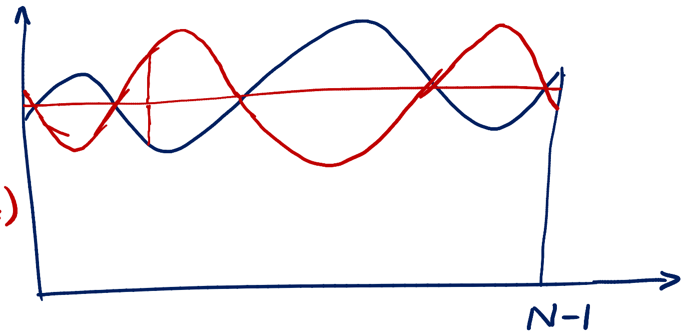
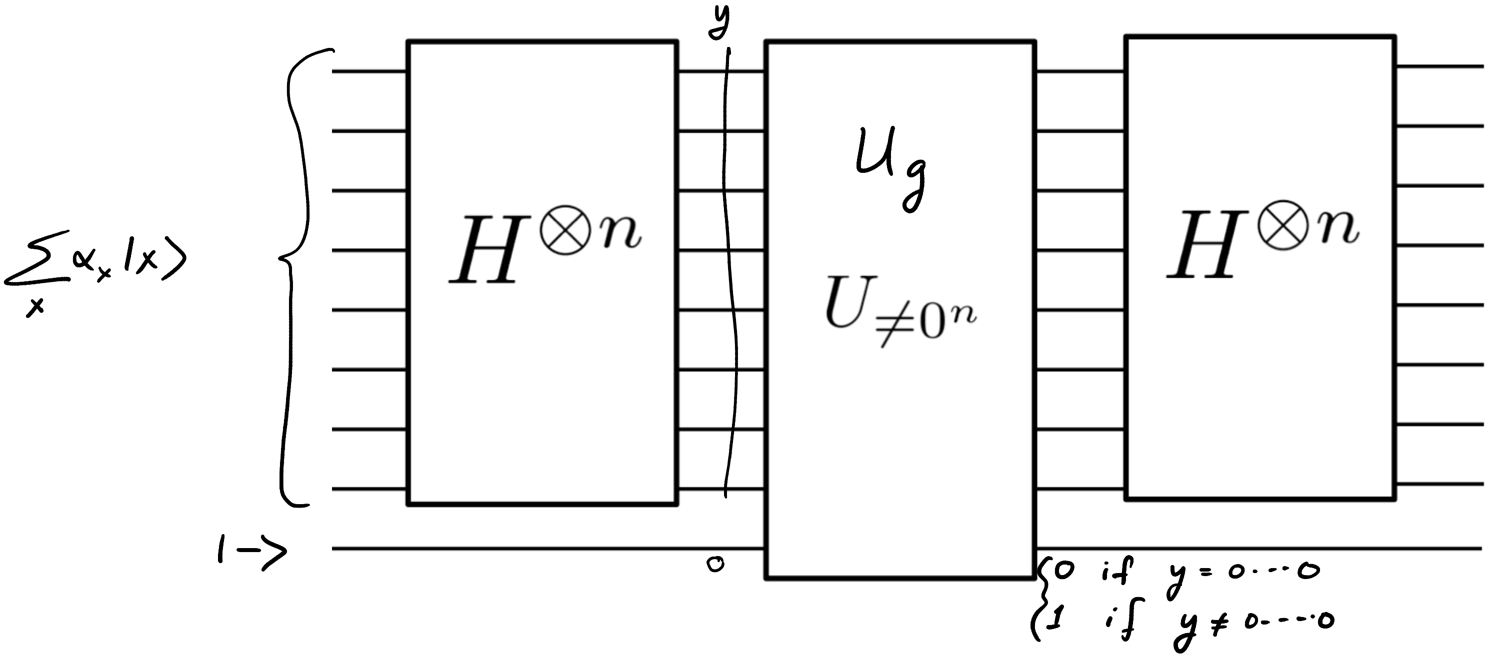


Inversion About Mean

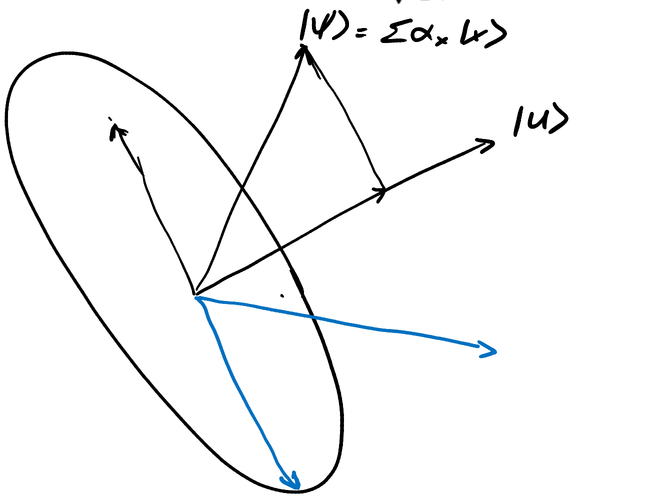
$g : \{0,1\}^n \to \{0,1\}$

$g(0\cdots0) = 0$

$g(y) = 1 \quad \text{if } y \neq 0\cdots0.$



Inversion about the mean is the same as doing reflection about $|u\rangle = \frac{1}{\sqrt{N}}\sum_x |x\rangle$



$$H^{\otimes n}\begin{bmatrix}1&&&\\&-1&&0\\&&\ddots&\\0&&&-1\end{bmatrix}H^{\otimes n}$$

$$= H^{\otimes n}\left(\begin{bmatrix}2&0&&\\&\ddots&&\\&&\ddots&\\&&&0\end{bmatrix} - I\right)H^{\otimes n}$$

$$= H^{\otimes n}\begin{bmatrix}2&0&&\\&\ddots&&\\&&&0\end{bmatrix}H^{\otimes n} - \underbrace{H^{\otimes n} I H^{\otimes n}}_{I}.$$

$$= \begin{bmatrix}2/\sqrt{N} & 0 & \cdots & 0\\ \vdots &&& \\ 2/\sqrt{N} & 0 & \cdots & 0\end{bmatrix} H^{\otimes n} - I$$

$N = 2^n$

$$= \begin{bmatrix}2/N & 2/N & \cdots & 2/N\\ \vdots & & & \vdots\\ 2/N & \cdots & & 2/N\end{bmatrix} - I = \begin{pmatrix}\frac{2}{N}-1 & \frac{2}{N} & \cdots & \frac{2}{N}\\ 2/N & \ddots & & \vdots\\ \vdots & & \ddots & 2/N\\ 2/N & \cdots & 2/N & \frac{2}{N}-1\end{pmatrix}$$

1) Transform $|u\rangle$ into $|0\cdots0\rangle$
2) Reflection about $|0\cdots0\rangle$
3) Transform $|0\cdots0\rangle$ into $|u\rangle$.

$$H^{\otimes n}\begin{bmatrix}1&&&\\&-1&&0\\&&\ddots&\\0&&&-1\end{bmatrix}H^{\otimes n}$$

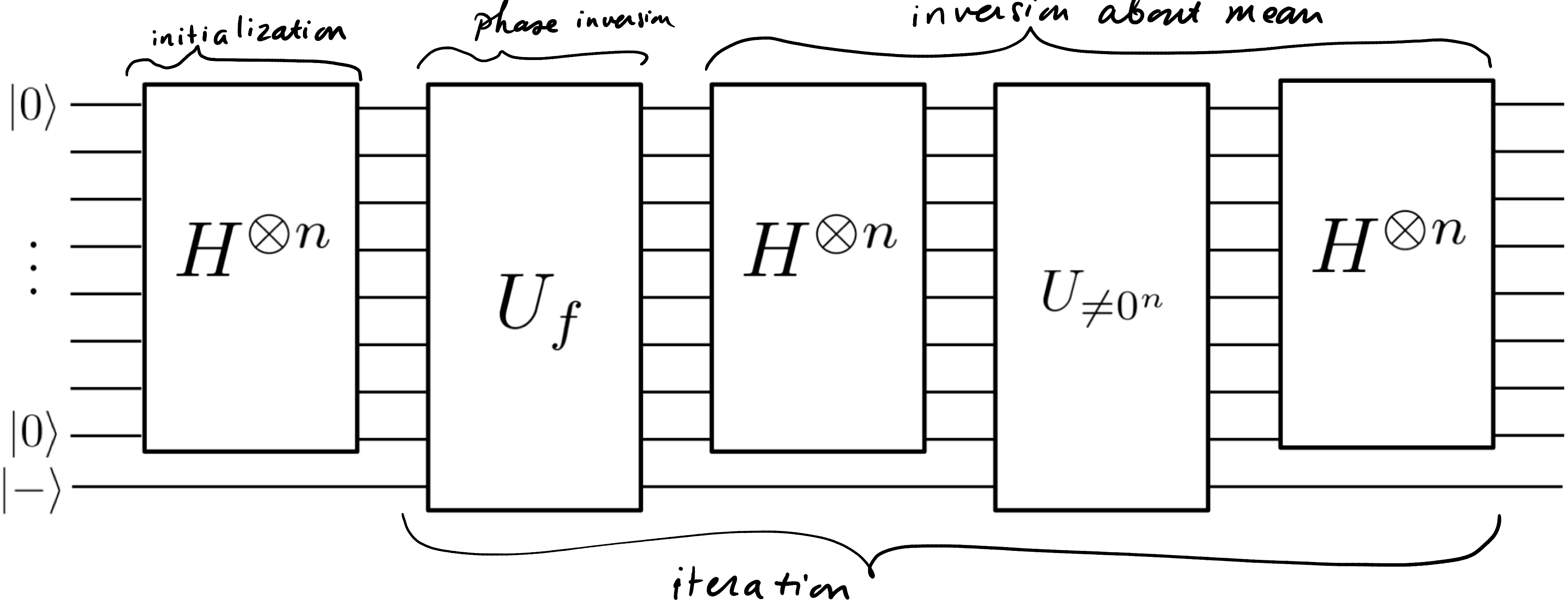
Inversion about the mean is the same as doing reflection about $|u\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle$

$$\begin{pmatrix} \frac{2}{N}-1 & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \ddots & & \vdots \\ \vdots & & \ddots & \frac{2}{N} \\ \frac{2}{N} & \cdots & \frac{2}{N} & \frac{2}{N}-1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_x \\ \vdots \\ \alpha_{N-1} \end{pmatrix} \longrightarrow \begin{pmatrix} \\ \frac{2}{N}\sum_{y=0}^{N-1}\alpha_y - \alpha_x \\ \\ \end{pmatrix} = 2\mu - \alpha_x$$

$$\frac{2}{N}\sum_y \alpha_y = 2\mu$$

Quantum Ckt for Grover's Alg.

$N = 2^n$



$O(\sqrt{N})$ iterations.