Video 3.1

Vijay Kumar and Ani Hsieh

Dynamics of Robot Arms

Vijay Kumar and Ani Hsieh
University of Pennsylvania

Lagrange's Equation of Motion

Lagrangian $\mathcal{L} = \boxed{\mathcal{K}} - \boxed{\mathcal{P}}$

Kinetic Energy

Potential Energy

1-DOF

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

n-DOF

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k \quad k = 1, \ldots, n$$

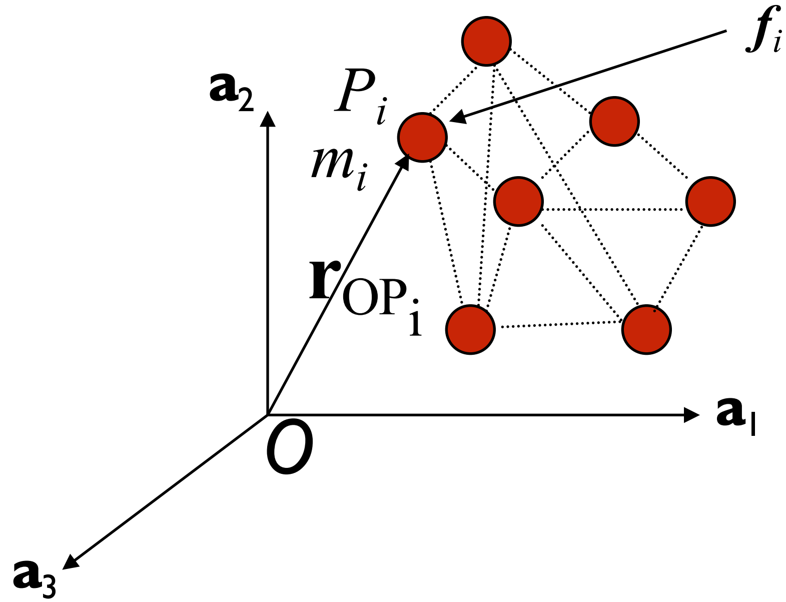
q, q_k Generalized Coordinates

τ, τ_k Generalized Forces

Motion of Systems of Particles

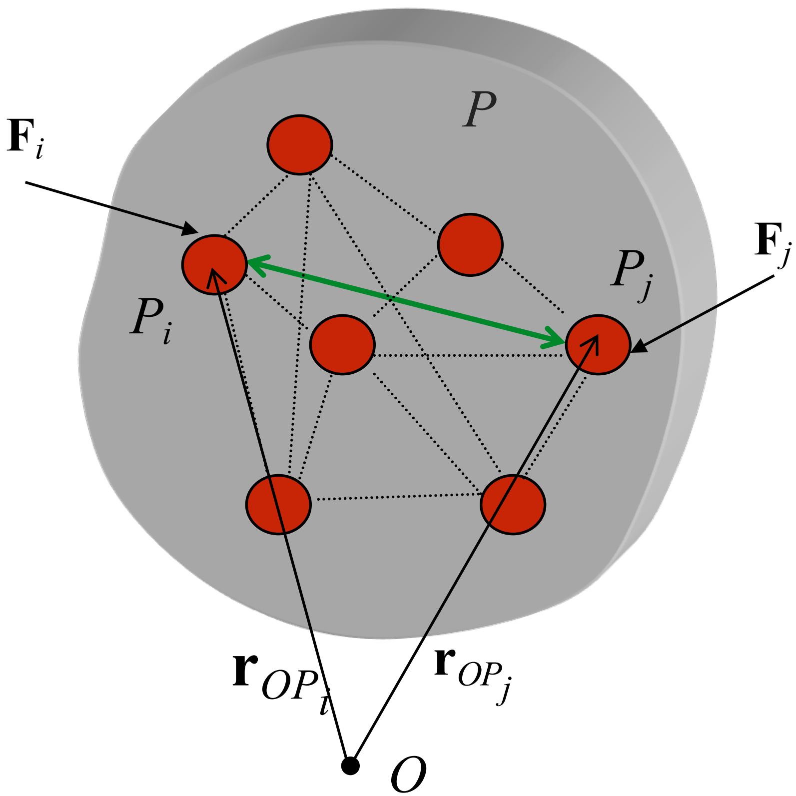
- Center of Mass

$$\mathbf{r}_C = \frac{1}{m}\sum_{i=1}^{k} m_i \mathbf{r}_{OP_i}$$



Newton's 2nd Law $$\mathbf{F} = \sum_{i=1}^{k} \mathbf{f}_i = m\mathbf{a}_C$$

Rigid Body as a System of Particles



- Constraints

$$\|\mathbf{r}_{OP_i} - \mathbf{r}_{OP_j}\| = l_{ij}$$

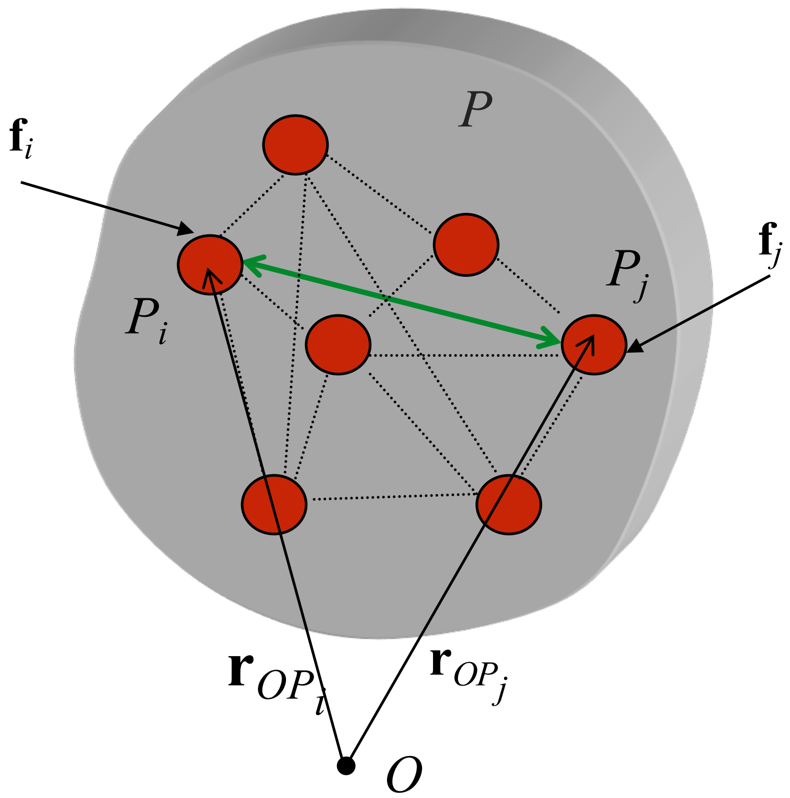
- *Holonomic* Constraints
  - Constraints on *position*

$$g_i(\mathbf{r}_{OP_1}, \ldots, \mathbf{r}_{OP_k}) = 0, \quad i = 1, \ldots l$$

Holonomic Constraints

- Given a system with *k* particles and *l* holonomic constraints
  - $DOF = k - l$
  - $n = k - l$ generalized coordinates
  - $\mathbf{r}_{OP_i} = \mathbf{r}_{OP_i}(q_1, \ldots, q_n), \quad i = 1, \ldots, k$
  - $q_1, \ldots, q_n$ are independent

Types of Displacements



- Actual
- Possible
- Virtual (or Admissible)

Video 3.2

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Classification of Forces

Newtonian **Lagrangian**

$$\mathbf{F}_i = \boxed{\sum_j \mathbf{f}_{ij}} + \boxed{\sum_k \mathbf{f}_i}$$

Internal vs External

Constraint vs Applied

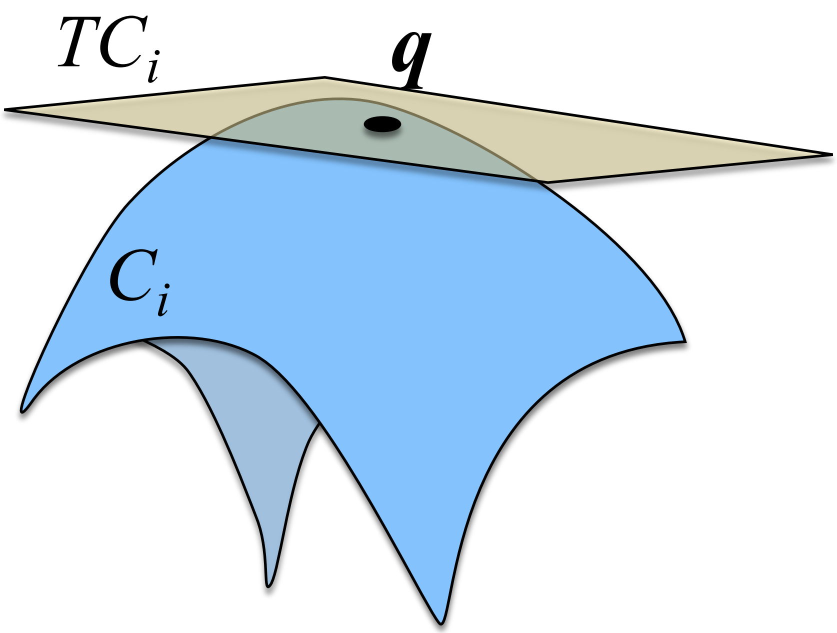
Applied Forces:

Any forces that are not constraint forces

D'Alembert's Principle

The totality of the constraint forces may be disregarded in the dynamics problem for a system of particles

D'Alembert's & Virtual Displacements



- C_i – Constraint Surface

  $$g_i(r_{OP_1}, \ldots, r_{OP_k}) = 0$$

- TC_i – Tangent space of C_i
- Virtual Displacements

  $\partial \mathbf{r}_{OP_i}$ satisfy:

  1. $g_i(r_{OP_1}, \ldots, r_{OP_k}) = 0$
  2. Eqn of Motion

Intuition for D'Alembert's (I)

From Newton's 2nd Law

$$\sum_{i=1}^{k} (m_i \ddot{\mathbf{r}}_i - \mathbf{f}_i) = \sum_{i=1}^{k} \mathbf{f}_i^a$$

$$\left[\sum_{i=1}^{k} (m_i \ddot{\mathbf{r}}_i - \mathbf{f}_i)\right]_{\perp} + \left[\sum_{i=1}^{k} (m_i \ddot{\mathbf{r}}_i - \mathbf{f}_i)\right]_{\|} = \left[\sum_{i=1}^{k} \mathbf{f}_i^a\right]_{\perp} + \left[\sum_{i=1}^{k} \mathbf{f}_i^a\right]_{\|}$$

$$\left[\sum_{i=1}^{k} (m_i \ddot{\mathbf{r}}_i - \mathbf{f}_i)\right]_{\perp} = \left[\sum_{i=1}^{k} \mathbf{f}_i^a\right]_{\perp}$$

$$\left[\sum_{i=1}^{k} (m_i \ddot{\mathbf{r}}_i - \mathbf{f}_i)\right]_{\|} = \left[\sum_{i=1}^{k} \mathbf{f}_i^a\right]_{\|}$$

Intuition for D'Alembert's (2)

By definition

$$\left[\sum_{i=1}^{k} \mathbf{f}_i^a\right]_{\perp} = 0 \quad \text{and} \quad \star = 0$$

And, $\left[\sum_{i=1}^{k} \mathbf{f}_i^a\right]_{\parallel} = 0$ b/c motion is constrained

$$\text{and} \quad \triangle = 0$$

D'Alembert's Principle

Alternative Form:

1. Tangent component of $\mathbf{f}_i$ are the only ones to contribute to the particle's acceleration

$$\sum_{i=1}^{k} m_i \ddot{\mathbf{r}}_i - (\mathbf{f}_i)_{\parallel} = 0$$

2. Normal components of $\mathbf{f}_i$ are in equilibrium w/ $\mathbf{f}_i^a$

$$\sum_{i=1}^{k} (\mathbf{f}_i)_{\perp} + \mathbf{f}_i^a = 0$$

Video 3.3

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Principle of Virtual Work

The totality of the constraint forces does no virtual work.

Virtual Work $\delta W = \mathbf{f} \cdot \delta \mathbf{r}$

$$\sum_{i=1}^{k} (m_i \ddot{\mathbf{r}}_i - \mathbf{f}_i) \cdot \delta \mathbf{r}_i = \sum_{i=1}^{k} \mathbf{f}_i^a \cdot \delta \mathbf{r}_i$$

By D'Alembert's Principle

$$\sum_{i=1}^{k} (m_i \ddot{\mathbf{r}}_i - \mathbf{f}_i) \cdot \delta \mathbf{r}_i = \sum_{i=1}^{k} \mathbf{f}_i^a \cdot \delta \mathbf{r}_i = 0$$

Lagrange's EOM for Systems of Particles (1)

System w/ k particles, l constraints, n = k-l DOF

Virtual Work $$\sum_{i=1}^{k} (m_i \ddot{\mathbf{r}}_i - \mathbf{f}_i) \cdot \delta \mathbf{r}_i = \sum_{i=1}^{k} \mathbf{f}_i^a \cdot \delta \mathbf{r}_i$$

$$\sum_{i=1}^{k} (m_i \ddot{\mathbf{r}}_i - \mathbf{f}_i) \cdot \delta \mathbf{r}_i = 0$$

$$\sum_{i=1}^{k} \mathbf{f}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{k} \sum_{j=1}^{n} \mathbf{f}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} \sum_{i=1}^{k} \mathbf{f}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} \psi_j \delta q_j$$

$$\psi_j = \sum_{i=1}^{k} \mathbf{f}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j}$$ j^{th} generalized force

Lagrange's EOM for Systems of Particles (2)

$$\sum_{i=1}^{k} m_i \ddot{\mathbf{r}}_i^T \delta \mathbf{r}_i = \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

$$\frac{d}{dt}\left[m_i \dot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j} \right] = m_i \ddot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j} + m_i \dot{\mathbf{r}}_i^T \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Note:

1) $\dot{\mathbf{r}}_i = \mathbf{v}_i = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j \Rightarrow \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}$

2) $\frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_{l=1}^{n} \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_l} \dot{q}_l = \frac{\partial}{\partial q_j} \sum_{l=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_l} \dot{q}_l = \frac{\partial \mathbf{v}_i}{\partial q_j}$

Lagrange's EOM for Systems of Particles (3)

$$\sum_{i=1}^{k} m_i \ddot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt} \left[m_i \mathbf{v}_i^T \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right] - m_i \mathbf{v}_i^T \frac{\partial \mathbf{v}_i}{\partial q_j} \right\}$$

Kinetic Energy $$K = \sum_{i=1}^{k} \frac{1}{2} m_i \mathbf{v}_i^T \mathbf{v}_i$$

$$\sum_{i=1}^{k} m_i \ddot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}$$

Lagrange's EOM for Systems of Particles (4)

$$\sum_{i=1}^{k} (m_i \ddot{\mathbf{r}}_i - \mathbf{f}_i) \cdot \delta \mathbf{r}_i = 0$$

$$\sum_{i=1}^{k} m_i \ddot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} \qquad \sum_{i=1}^{k} \mathbf{f}_i \cdot \delta \mathbf{r}_i = \sum_{j=1}^{n} \psi_j \delta q_j$$

And if $\psi_j = -\frac{\partial P}{\partial q_j} + \tau_j$

P - Potential Energy

τ_j - Generalized Applied Forces

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j}$$

Summary

$$\sum_{i=1}^{k} (m_i \ddot{\mathbf{r}}_i - \mathbf{f}_i) \cdot \delta \mathbf{r}_i = 0 \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j$$

- $\mathbf{f}_i$ - vector in 3D
- Virtual work $\delta W = \mathbf{f} \cdot \delta \mathbf{q} = \sum_{j=1}^{n} f_j \delta q_j$
- f_j - component in the direction of $\hat{\mathbf{e}}_{q_j}$

DO virtual work vs. DO NOT

ONLINE LEARNING

Video 3.4

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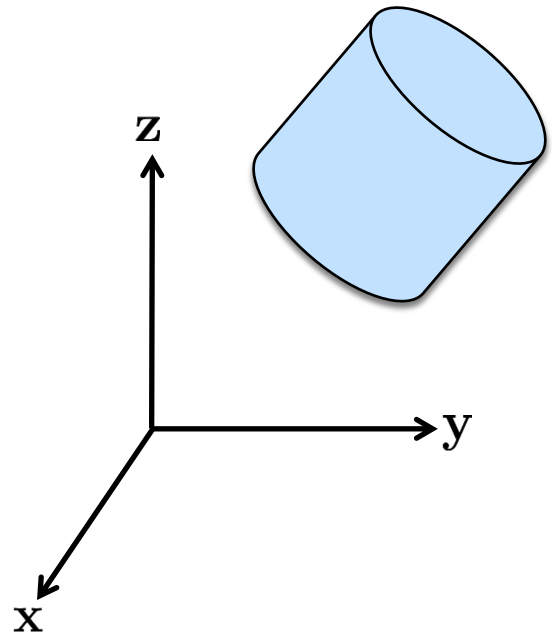
Potential Energy

$$P = m\mathbf{g}^T\mathbf{r}_C$$



Kinetic Energy

Kinetic energy of a rigid body consists of two parts

$$\mathcal{K} = \boxed{\frac{1}{2} m \mathbf{v}^T \mathbf{v}} + \boxed{\frac{1}{2} \omega^T \mathcal{I} \omega}$$

Translational Rotational

Inertia Tensor $\mathcal{I} = \mathbf{R} \mathbf{I} \mathbf{R}^T$

Inertia Tensor

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Principal Moments

Cross Products of Inertia

- 3x3 matrix
- Symmetric matrix

$$I_{xy} = I_{yx}, I_{xz} = I_{zx}, I_{yz} = I_{zy}$$

Let $\rho(x, y, z)$ denote the mass density

Principal Moments of Inertia

$$\begin{cases} I_{xx} = \int\int\int (y^2 + z^2)\rho(x, y, z)dxdydz \\ I_{yy} = \int\int\int (x^2 + z^2)\rho(x, y, z)dxdydz \\ I_{zz} = \int\int\int (x^2 + y^2)\rho(x, y, z)dxdydz \end{cases}$$

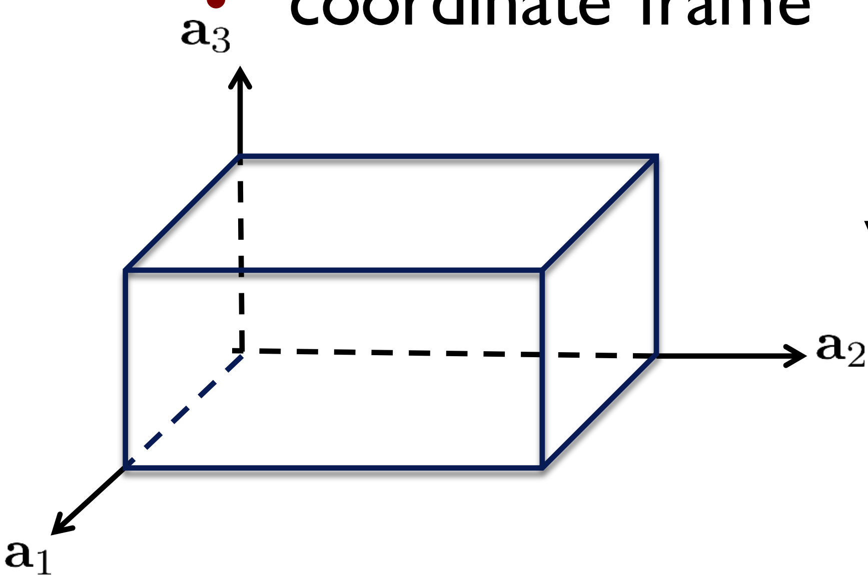
Cross Products of Inertia

$$\begin{cases} I_{xy} = I_{yx} = -\int\int\int xy\rho(x, y, z)dxdydz \\ I_{xz} = I_{zx} = -\int\int\int xz\rho(x, y, z)dxdydz \\ I_{yz} = I_{zy} = -\int\int\int yz\rho(x, y, z)dxdydz \end{cases}$$

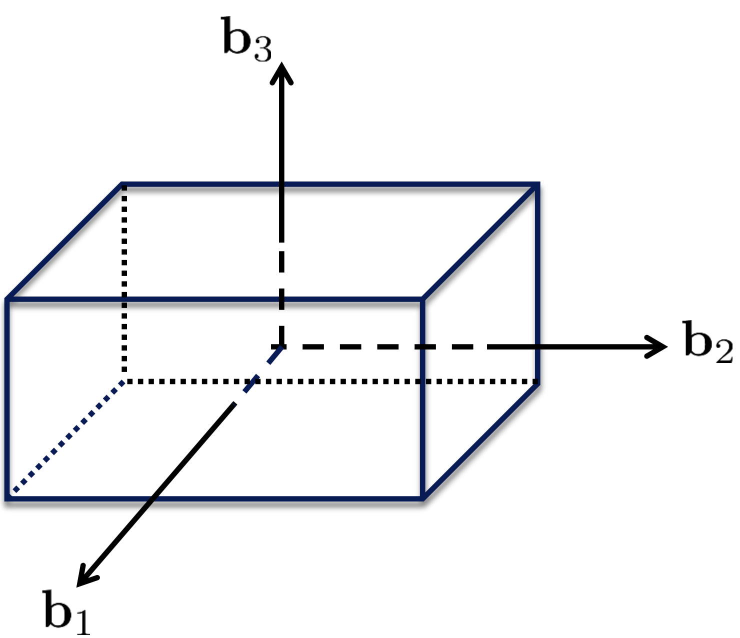
Remarks

Inertia tensor depends on

- reference point
- coordinate frame



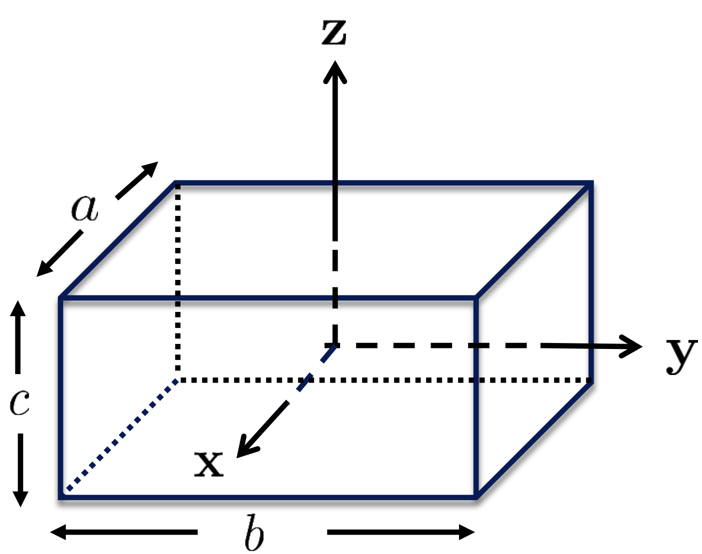
VS



Example

Compute the inertia tensor of the block with the given dimensions.

Assume $\rho(x, y, z)$ is constant.



Video 3.5
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Potential Energy for n-Link Robot

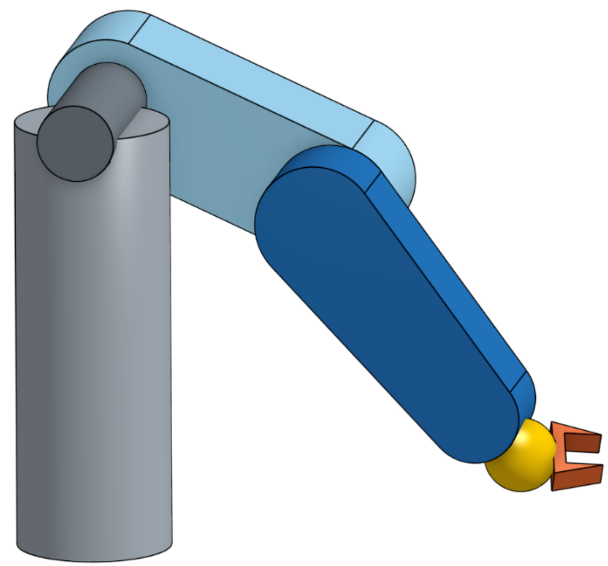
- 1-Link Robot

$$P = m\mathbf{g}^T\mathbf{r}_C$$

- n-Link Robot

$$P_i = m_i\mathbf{g}^T\mathbf{r}_{C_i}$$

$$P = \sum_{i=1}^{n} m_i\mathbf{g}^T\mathbf{r}_{C_i}$$



Kinetic Energy for n-Link Robot (1)

- 1-Link Robot

$$\mathcal{K} = \frac{1}{2} m \mathbf{v}^T \mathbf{v} + \frac{1}{2} \omega^T \mathcal{I} \omega$$

- n-Link Robot

$$\mathcal{K} = \sum_{i=1}^{n} \left\{ \frac{1}{2} m \mathbf{v}_i{}^T \mathbf{v}_i + \frac{1}{2} \omega_i^T \mathcal{I}_i \omega_i \right\}$$

Review of the Jacobian

$$\mathbf{q} \in \mathbb{R}^n$$

$$\mathbf{f}(\mathbf{q}) \in \mathbb{R}^m \qquad \mathbf{f}: \mathbb{R}^n \to \mathbb{R}^m$$

$$\mathbf{J} = \left[\frac{\partial \mathbf{f}}{\partial q_1} \quad \cdots \quad \frac{\delta \mathbf{f}}{\partial q_n}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix}$$

$$\mathbf{J}_{ij} = \frac{\partial f_i}{\partial q_j}$$

Kinetic Energy of n-Link Robot (2)

$$\mathbf{v}_i = \mathbf{J}_{v_i}(\mathbf{q})\dot{\mathbf{q}} \qquad \omega_i = \mathbf{J}_{\omega_i}(\mathbf{q})\dot{\mathbf{q}}$$

$$\mathcal{K} = \frac{1}{2}\dot{\mathbf{q}}^T \left[\sum_{i=1}^{n} m_i \mathbf{J}_{v_i}^T \mathbf{J}_{v_i} + \mathbf{J}_{\omega_i}^T \mathbf{R}_i I_i \mathbf{R}_i^T \mathbf{J}_{\omega_i} \right] \dot{\mathbf{q}}$$

$$\mathbf{D}(\mathbf{q}) = \left[\sum_{i=1}^{n} m_i \mathbf{J}_{v_i}^T \mathbf{J}_{v_i} + \mathbf{J}_{\omega_i}^T \mathbf{R}_i I_i \mathbf{R}_i^T \mathbf{J}_{\omega_i} \right]$$

Euler-Lagrange EOM for n-Link Robot (I)

Assumptions:

- $\mathcal{K}$ is quadratic function of $\dot{\mathbf{q}}$
- $\mathcal{P} = \mathcal{P}(\mathbf{q})$ and independent of $\dot{\mathbf{q}}$

$$\mathcal{L} = \mathcal{K} - \mathcal{P} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{D}(\mathbf{q})\dot{\mathbf{q}} - \mathcal{P}(\mathbf{q})$$

$$= \frac{1}{2}\sum_{i,j} d_{ij}(\mathbf{q})\dot{q}_i\dot{q}_j - \mathcal{P}(\mathbf{q})$$

Euler-Lagrange EOM for n-Link Robot (2)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k$$

$$\mathcal{L} = \frac{1}{2}\sum_{i,j} d_{ij}(\mathbf{q})\dot{q}_i\dot{q}_j - \mathcal{P}(\mathbf{q})$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_j d_{kj}\dot{q}_j$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

Euler-Lagrange EOM for n-Link Robot (3)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k \qquad \mathcal{L} = \frac{1}{2}\sum_{i,j} d_{ij}(\mathbf{q})\dot{q}_i\dot{q}_j - \mathcal{P}(\mathbf{q})$$

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial \mathcal{P}}{\partial q_k}$$

$$\sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j - \sum_{i,j}\frac{1}{2}\left\{\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$$

Euler-Lagrange EOM for n-Link Robot (4)

$$\sum_{j=1}^{n} d_{kj}(\mathbf{q})\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}\dot{q}_i\dot{q}_j + g_k(\mathbf{q}) = \tau_k$$

$$k = 1, \ldots, n$$

$$g_k = \frac{\partial \mathcal{P}}{\partial q_k}$$

Christoffel Symbols $\quad c_{ijk} \equiv \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$

In matrix form $\quad \mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau$

Skew Symmetry

$$\boxed{\mathbf{D}(\mathbf{q})}\ddot{\mathbf{q}} + \boxed{\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau$$

$$\mathbf{D}(\mathbf{q}) = \left[\sum_{i=1}^{n} m_i \mathbf{J}_{v_i}^T \mathbf{J}_{v_i} + \mathbf{J}_{\omega_i}^T \mathbf{R}_i I_i \mathbf{R}_i^T \mathbf{J}_{\omega_i}\right]$$

$$\begin{aligned} c_{jk} &= \sum_{i=1}^{n} c_{ijk}(\mathbf{q})\dot{q}_i \\ &= \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i \end{aligned}$$

$$[\dot{\mathbf{D}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})] = -[\dot{\mathbf{D}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})]^T$$

Passivity

$$\int_0^T \dot{\mathbf{q}}^T \tau(\varepsilon) d\varepsilon \geq -\beta$$

- Power = Force x Velocity
- Energy dissipated over finite time is bounded
- Important for Controls

Bounds on D(q)

- $\lambda_i(\mathbf{q})$ - eigenvalue of $\mathbf{D}(\mathbf{q})$
- $0 \leq \lambda_1(\mathbf{q}) \leq \ldots \leq \lambda_n(\mathbf{q})$

$$\lambda_1(\mathbf{q})\mathbf{I} \leq \mathbf{D}(\mathbf{q}) \leq \lambda_n(\mathbf{q})\mathbf{I}$$

Linearity in the Parameters

System Parameters:

- Mass, moments of inertia, lengths, etc.

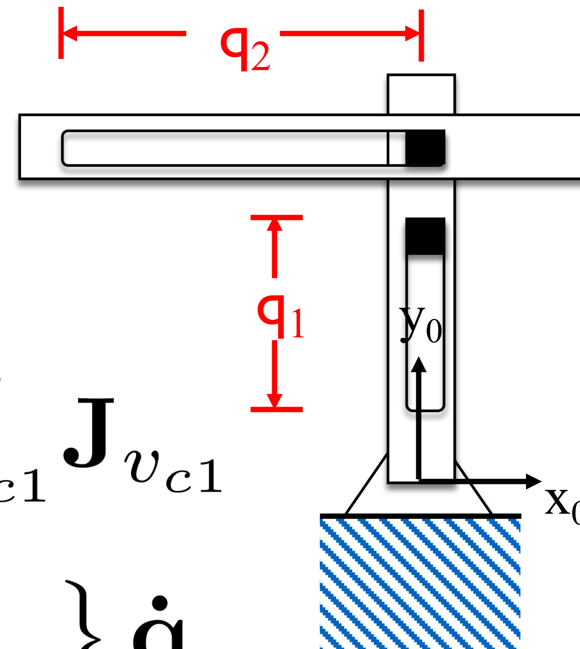
$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau$$

$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\mathbf{\Theta}$$

2-Link Cartesian Manipulator (I)

$$\mathcal{L} = \mathcal{K} - \mathcal{P}$$

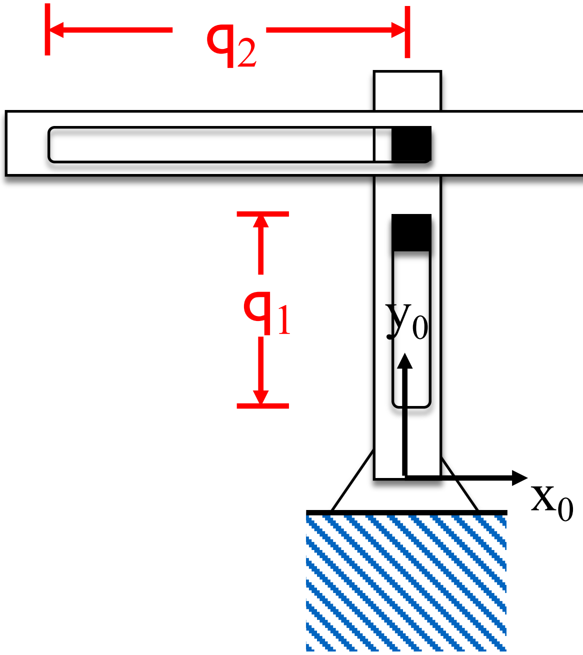
$$\mathcal{P} = g(m_1 + m_2)q_1$$



$$\mathcal{K} = \frac{1}{2}\dot{\mathbf{q}}^T \left\{ m_1 \mathbf{J}_{v_{c1}}^T \mathbf{J}_{v_{c1}} + m_2 \mathbf{J}_{v_{c2}}^T \mathbf{J}_{v_{c2}} \right\} \dot{\mathbf{q}}$$

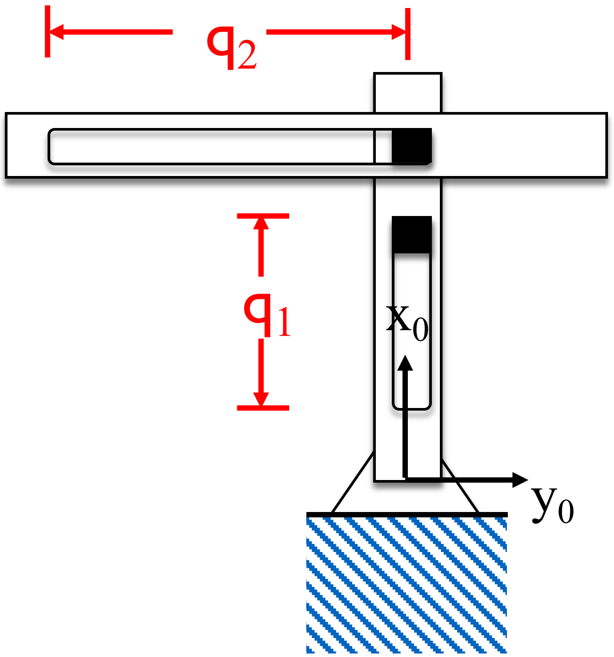
2-Link Cartesian Manipulator (2)

$$\mathcal{K} = \frac{1}{2}\dot{\mathbf{q}}^T \left\{ m_1 \mathbf{J}_{v_{c1}}^T \mathbf{J}_{v_{c1}} + m_2 \mathbf{J}_{v_{c2}}^T \mathbf{J}_{v_{c2}} \right\} \dot{\mathbf{q}}$$



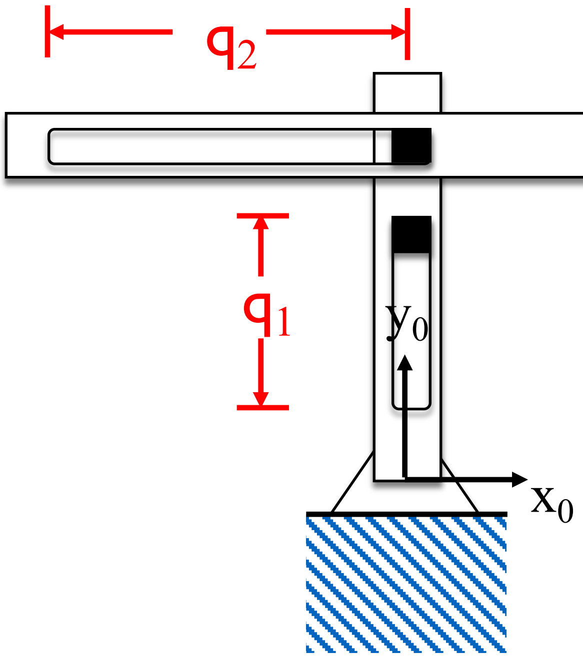
2-Link Cartesian Manipulator (3)

$$\mathcal{K} = \frac{1}{2}\dot{\mathbf{q}}^T \left\{ m_1 \mathbf{J}_{v_{c1}}^T \mathbf{J}_{v_{c1}} + m_2 \mathbf{J}_{v_{c2}}^T \mathbf{J}_{v_{c2}} \right\} \dot{\mathbf{q}}$$



2-Link Cartesian Manipulator (4)

$$\sum_{j=1}^{n} d_{kj}(\mathbf{q})\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}\dot{q}_i\dot{q}_j + g_k(\mathbf{q}) = \tau_k$$



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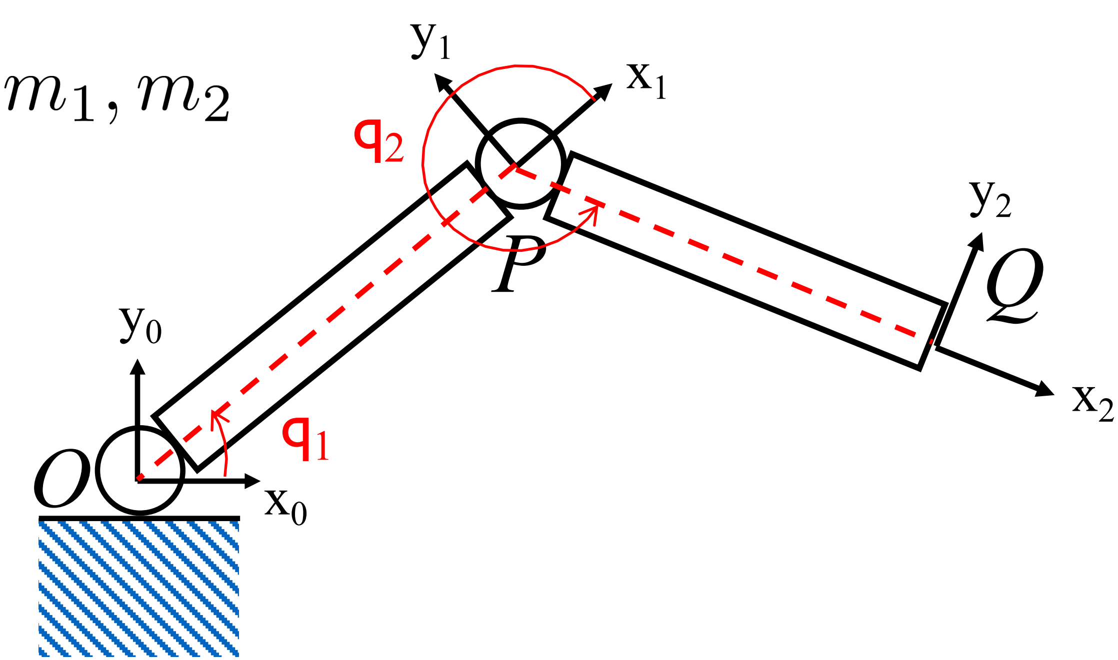
2-Link Planar Manipulator (I)

System parameters:

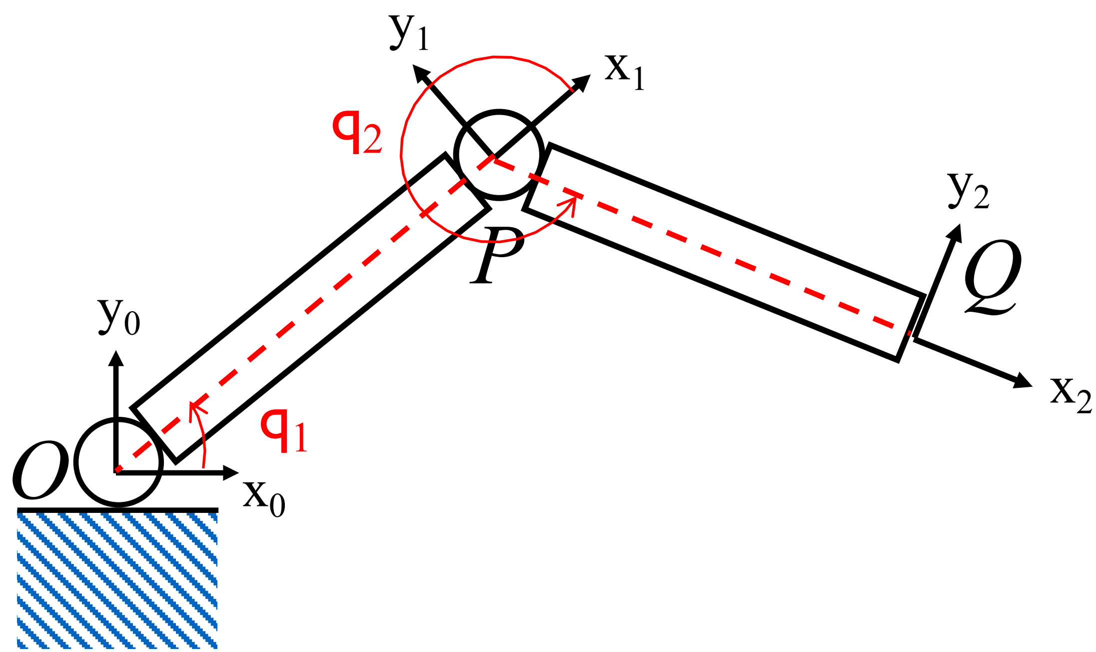
- Link lengths a_1, a_2
- Link center of mass location a_{c_1}, a_{c_2}
- Link masses m_1, m_2



2-Link Planar Manipulator (2)

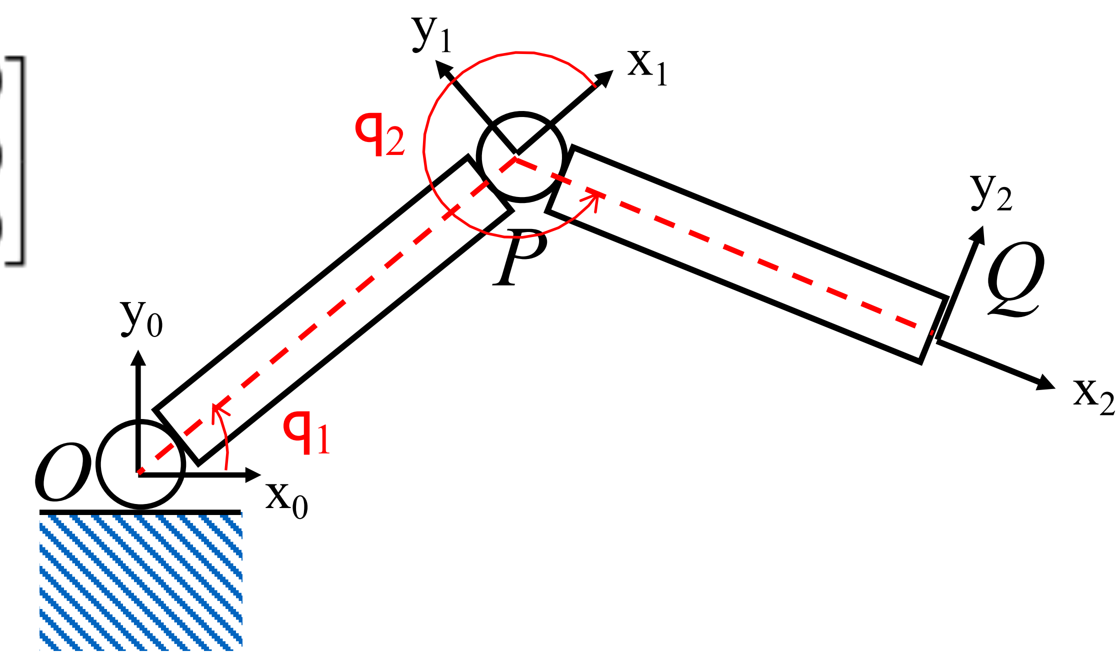
Recall $\mathbf{r}_{C_{OP}} = a_{c_1}\cos(q_1)\mathbf{x}_0 + a_{c_1}\sin(q_1)\mathbf{y}_0$

$$\begin{aligned}\mathbf{r}_{C_{PQ}} &= (a_1\cos(q_1) + a_{c_2}\cos(q_1+q_2))\,\mathbf{x}_0 \\ &\quad + (a_1\sin(q_1) + a_{c_2}\sin(q_1+q_2))\,\mathbf{y}_0\end{aligned}$$



2-Link Planar Manipulator (3)

$$\mathbf{J}_{v_{C1}} = \begin{bmatrix} -a_{C_1}\sin(q_1) & 0 \\ a_{C_1}\cos(q_1) & 0 \\ 0 & 0 \end{bmatrix}$$

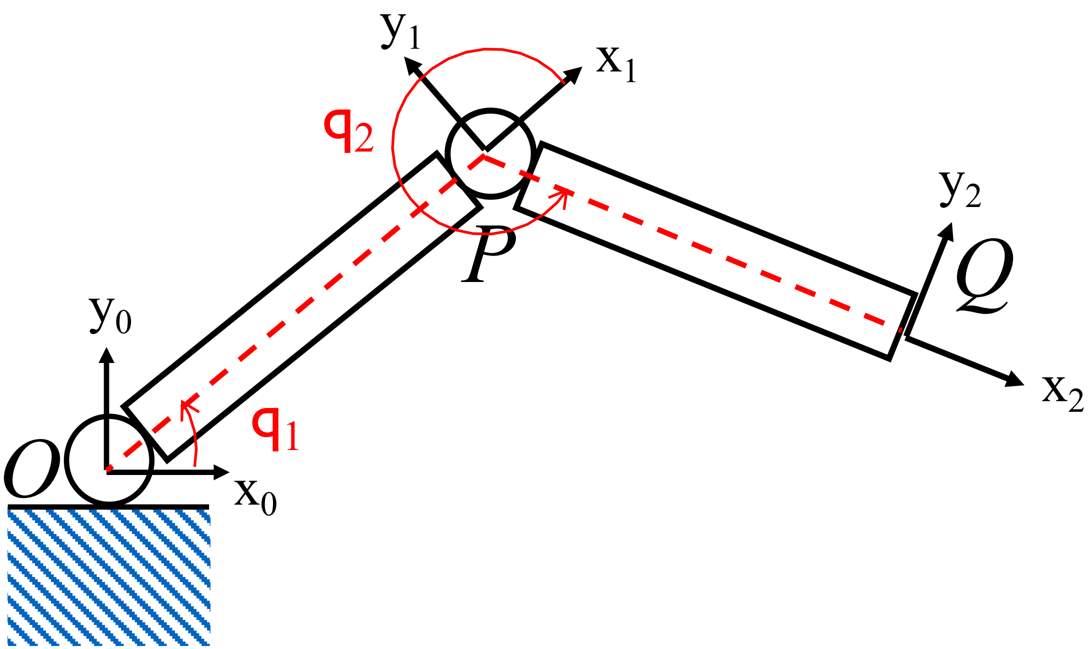


$$\mathbf{J}_{v_{C_2}} = \begin{bmatrix} -a_1\sin(q_1) - a_{C_2}\sin(q_1+q_2) & -a_{C_2}\sin(q_1+q_2) \\ a_1\cos(q_1) + a_{C_2}\cos(q_1+q_2) & a_{C_2}\cos(q_1+q_2) \\ 0 & 0 \end{bmatrix}$$

2-Link Planar Manipulator (4)

Kinetic Energy = Translational + Rotational

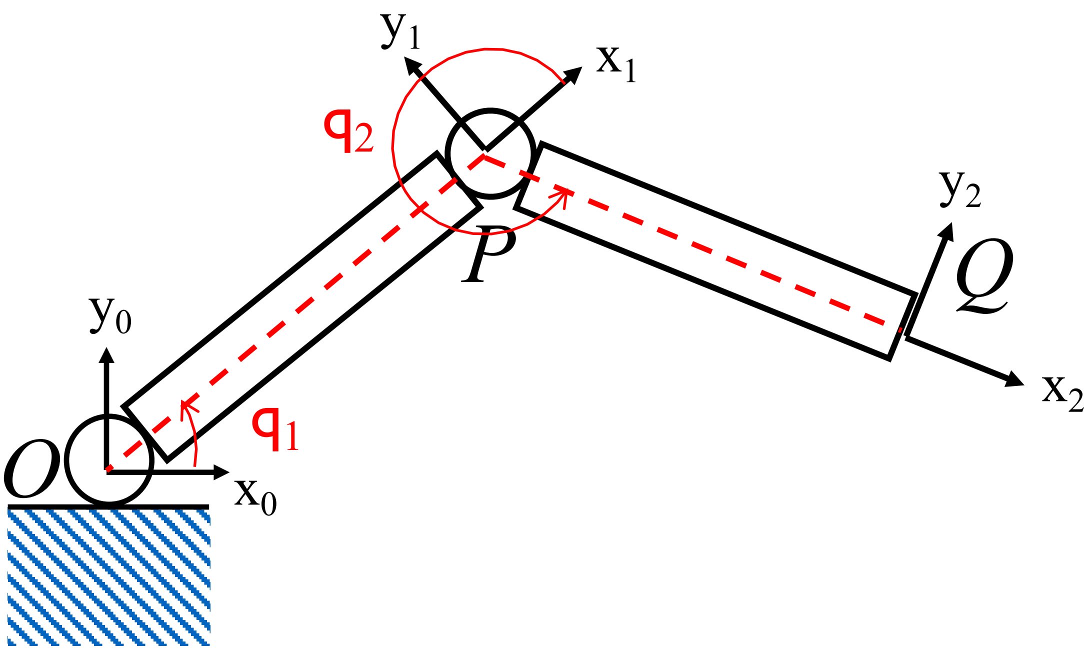
Translational



2-Link Planar Manipulator (5)

Kinetic Energy = Translational + Rotational

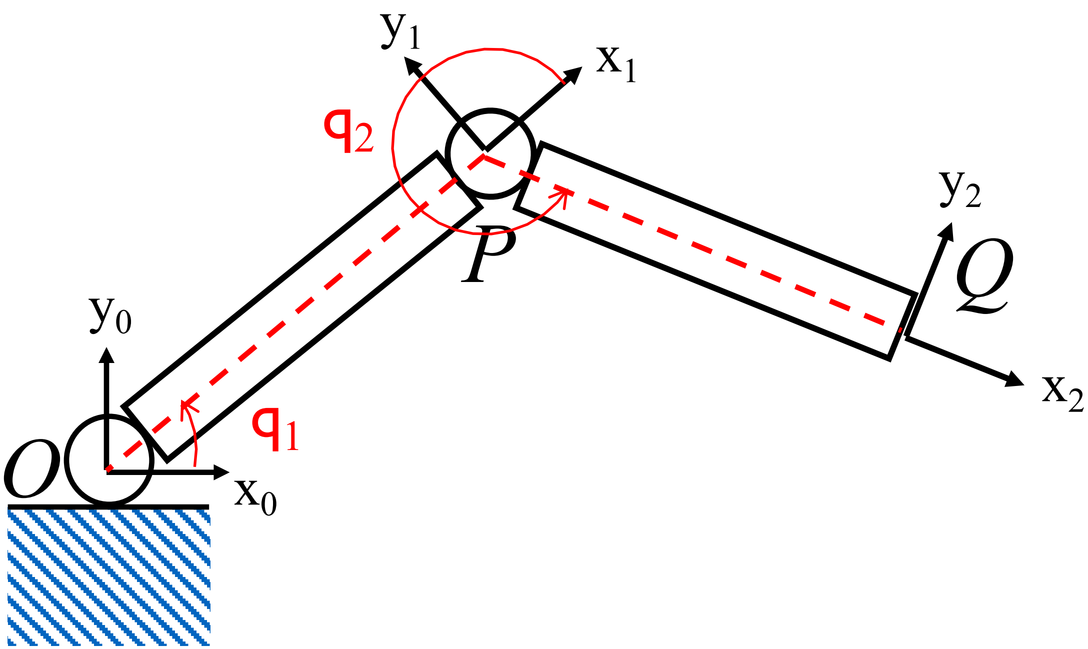
Rotational



2-Link Planar Manipulator (6)

Kinetic Energy = Translational + Rotational

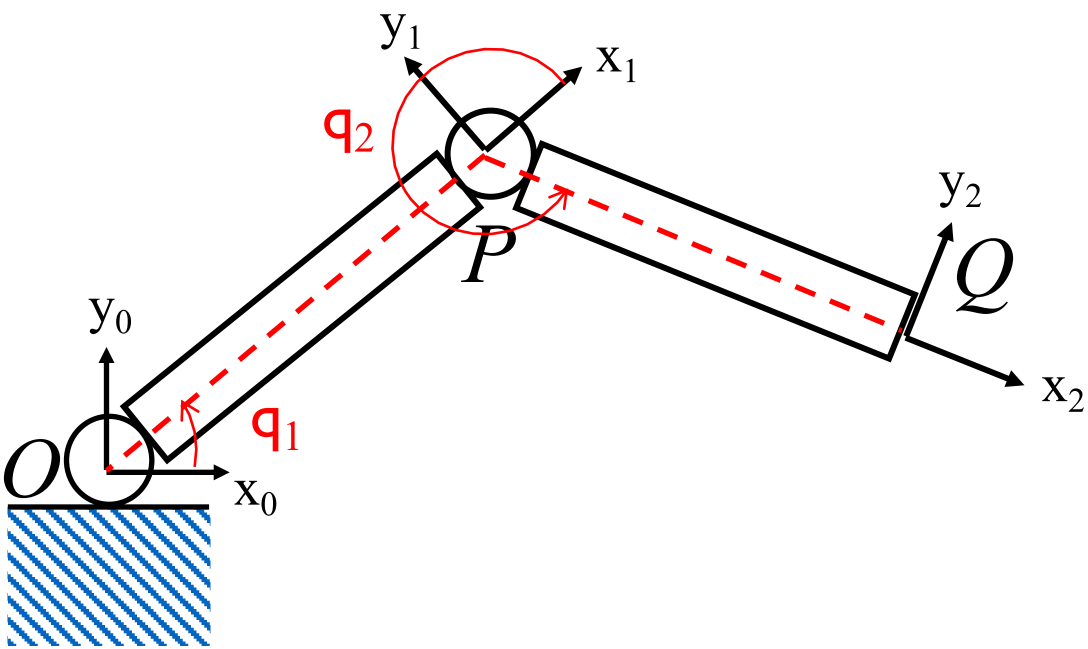
Rotational



2-Link Planar Manipulator (7)

Kinetic Energy = Translational + Rotational

Rotational



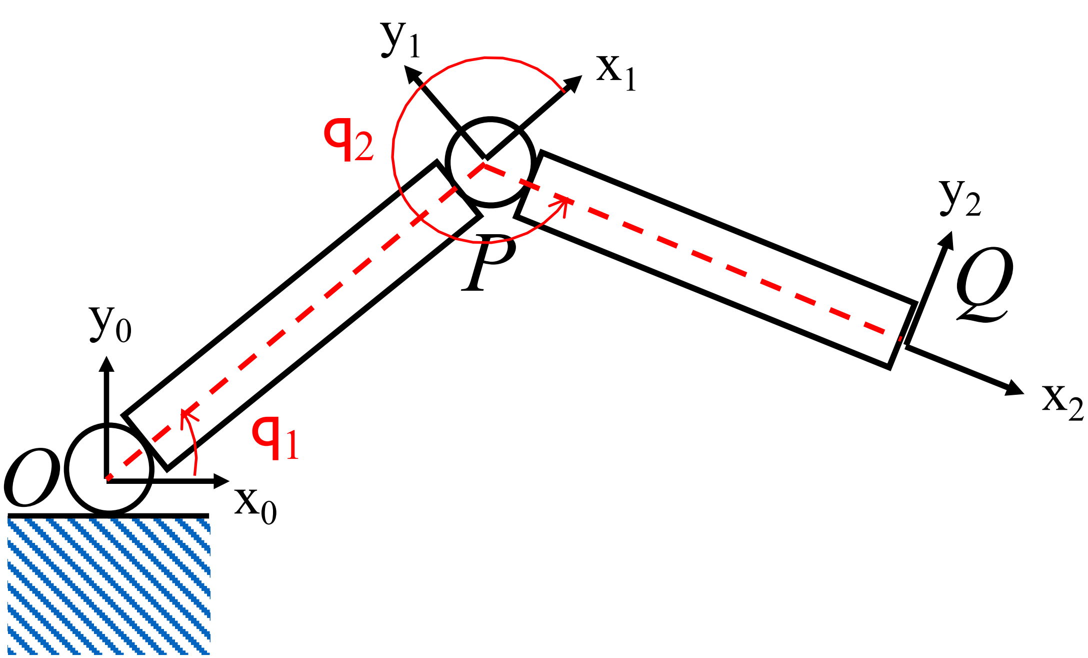
Video 3.7

Vijay Kumar and Ani Hsieh

2-Link Planar Manipulator (8)

Kinetic Energy = Translational + Rotational

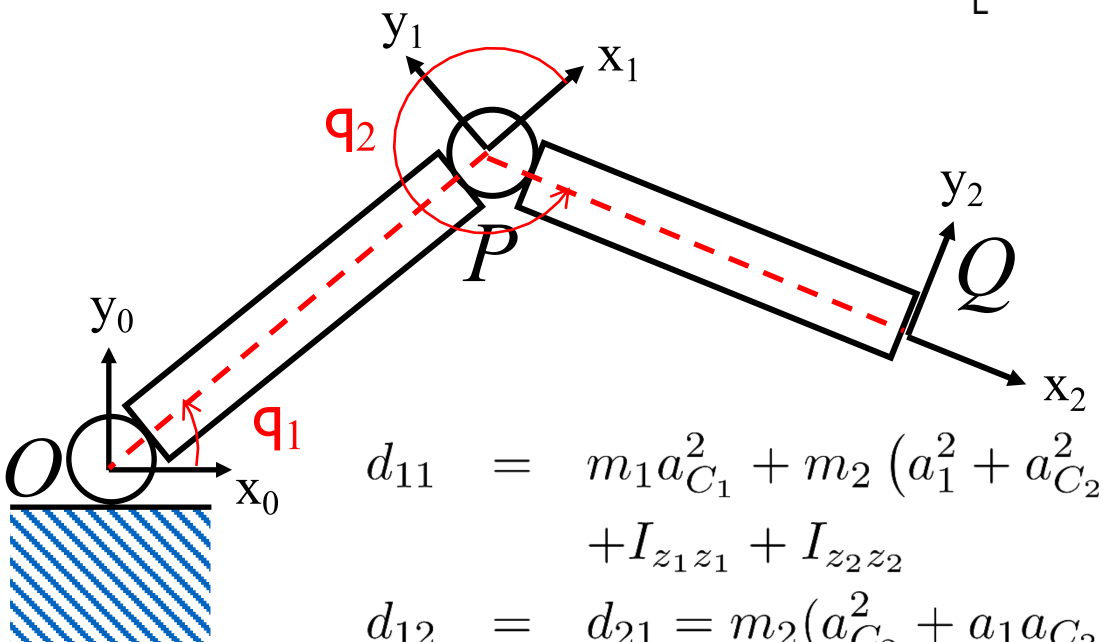
Rotational



2-Link Planar Manipulator (6)

$$\frac{1}{2}\dot{\mathbf{q}}^T\left\{I_{z_1z_1}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + I_{z_2z_2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\}\dot{\mathbf{q}}$$

$$\mathbf{D}(\mathbf{q}) = m_1\mathbf{J}_{v_{C_1}}^T\mathbf{J}_{v_{C_1}} + m_2\mathbf{J}_{v_{C_2}}^T\mathbf{J}_{v_{C_2}} + \begin{bmatrix} I_{z_1z_1}+I_{z_2z_2} & I_{z_2z_2}\\ I_{z_2z_2} & I_{z_2z_2}\end{bmatrix}$$



$$\begin{aligned} d_{11} &= m_1a_{C_1}^2 + m_2\left(a_1^2 + a_{C_2}^2 + 2a_1a_{C_2}\cos(q_2)\right)\\ &\quad + I_{z_1z_1} + I_{z_2z_2}\\ d_{12} &= d_{21} = m_2(a_{C_2}^2 + a_1a_{C_2}\cos(q_2) + I_{z_2z_2}\\ d_{22} &= m_2a_{C_2}^2 + I_{z_2z_2}\end{aligned}$$

2-Link Planar Manipulator (7)

$$
\begin{aligned}
d_{11} &= m_1 a_{C_1}^2 + m_2 \left(a_1^2 + a_{C_2}^2 + 2 a_1 a_{C_2} \cos(q_2) \right) + I_{z_1 z_1} + I_{z_2 z_2} \\
d_{12} &= d_{21} = m_2 (a_{C_2}^2 + a_1 a_{C_2} \cos(q_2) + I_{z_2 z_2} \\
d_{22} &= m_2 a_{C_2}^2 + I_{z_2 z_2}
\end{aligned}
$$

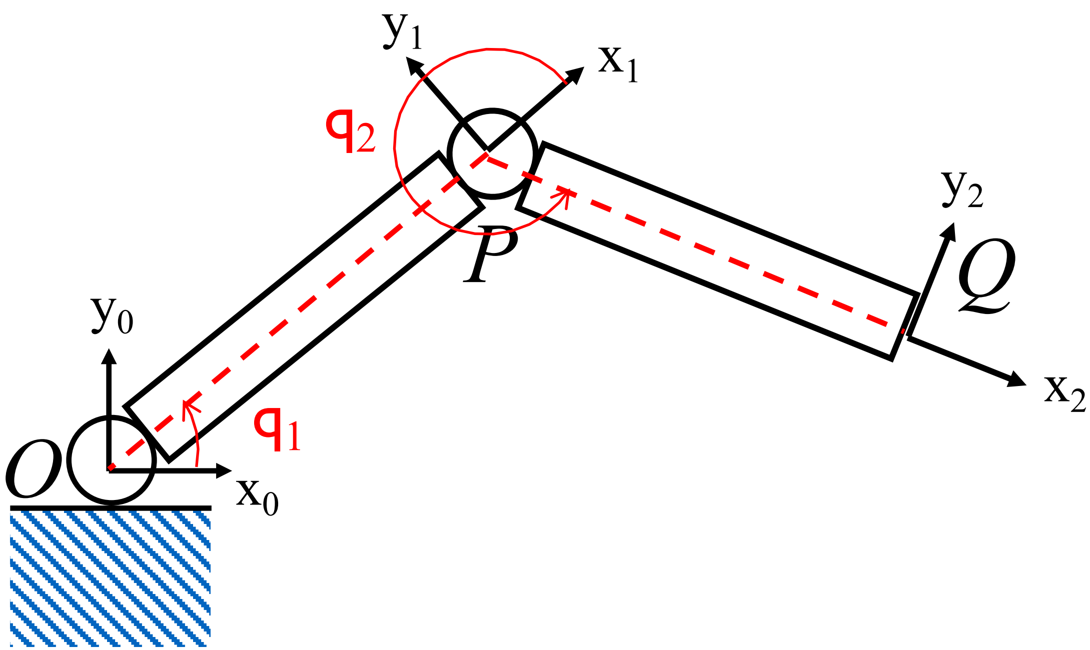
Christoffel Symbols

$$
\begin{aligned}
c_{111} &= \frac{1}{2} \frac{\partial d_{11}}{\partial q_1} \\
c_{121} &= c_{211} = \frac{1}{2} \frac{\partial d_{11}}{\partial q_2} \\
c_{221} &= \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2} \frac{\partial d_{22}}{\partial q_1}
\end{aligned}
$$

$$
\begin{aligned}
c_{112} &= \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2} \frac{\partial d_{11}}{\partial q_2} \\
c_{122} &= c_{212} = \frac{1}{2} \frac{\partial d_{22}}{\partial q_1} \\
c_{222} &= \frac{1}{2} \frac{\partial d_{22}}{\partial q_2}
\end{aligned}
$$

2-Link Planar Manipulator (8)

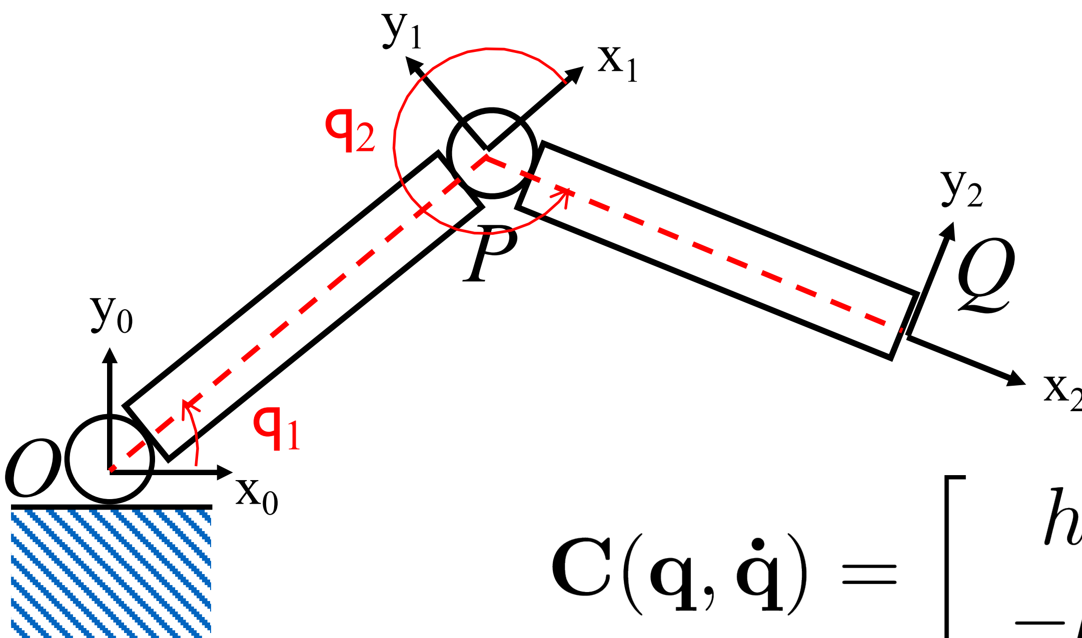
Potential Energy



2-Link Planar Manipulator (9)

Putting it all together

$$\begin{aligned} d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + g_1 &= \tau_1 \\ d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 &= \tau_2 \end{aligned}$$



$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} h\dot{q}_2 & h\dot{q}_2 + h\dot{q}_1 \\ -h\dot{q}_1 & 0 \end{bmatrix}$$

Newton-Euler vs. Euler-Lagrange

- N-E: Newton's Laws of Motion
- N-E: Explicit accounting for constraints
- N-E: Explicit accounting of the reference frame

- E-L: D'Alembert's Principle + Principle of Virtual Work
- E-L: Invariant under point transformations

Summary

- Lagrangian $\mathcal{L} = \mathcal{K} - \mathcal{P}$
- D'Alembert's Principle + Principle of Virtual Work
- Euler-Lagrange EOM $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k$

  $k = 1, \ldots, n$

- Properties of the E-L EOM
- Examples: 2 Link Planar Manipulators