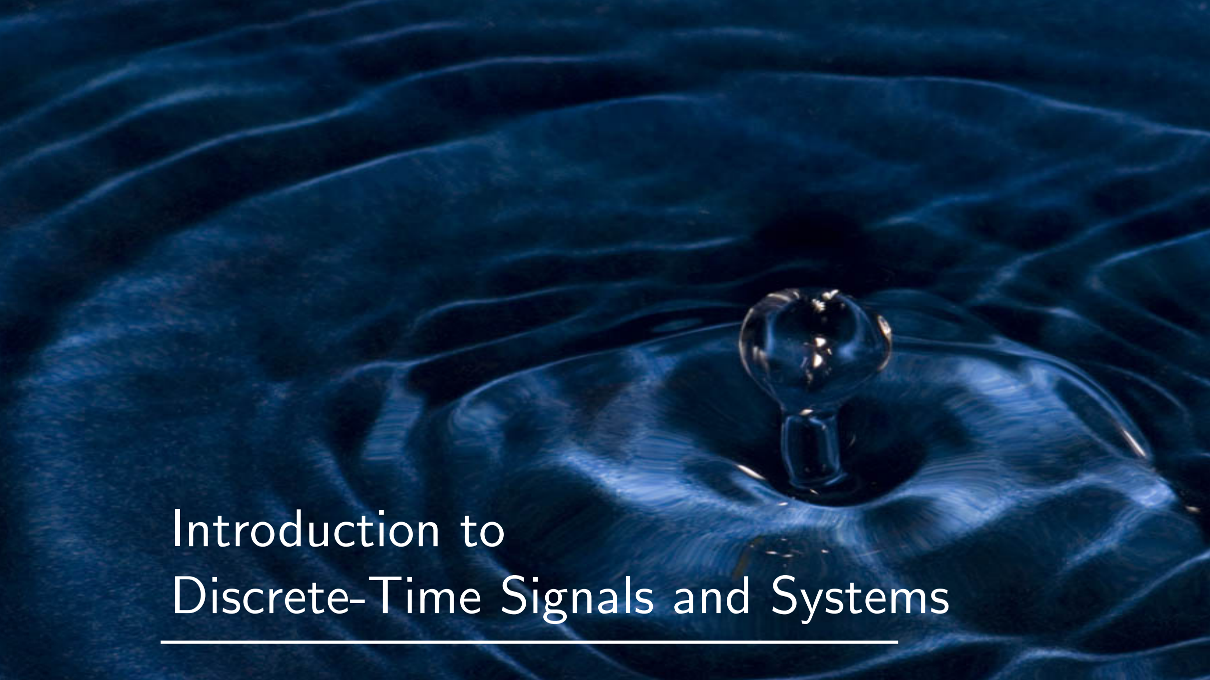


Introduction to Discrete-Time Signals and Systems

Welcome to Discrete Time Signals and Systems

- This is an introductory course on **signal processing** that studies **signals** and **systems**

> DEFINITION
>
> **Signal** (n): A detectable physical quantity ... by which messages or information can be transmitted (Merriam-Webster)

- Signals carry **information**

- Examples:
  - Speech signals transmit language via acoustic waves
  - Radar signals transmit the position and velocity of targets via electromagnetic waves
  - Electrophysiology signals transmit information about processes inside the body
  - Financial signals transmit information about events in the economy

Welcome to Discrete Time Signals and Systems

- **Systems** manipulate the information carried by signals

DEFINITION

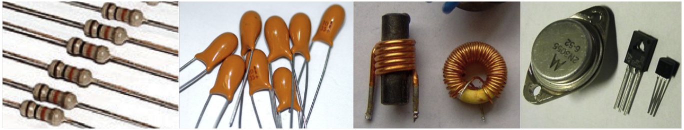
Signal processing involves the theory and application of

- filtering, coding, transmitting, estimating, detecting, analyzing, recognizing, synthesizing, recording, and reproducing signals by digital or analog devices or techniques
- where signal includes audio, video, speech, image, communication, geophysical, sonar, radar, medical, musical, and other signals

(IEEE Signal Processing Society Constitutional Amendment, 1994)

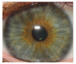
Signal Processing

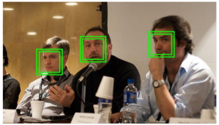
- Signal processing has traditionally been a part of electrical and computer engineering
- But now expands into applied mathematics, statistics, computer science, geophysics, and host of application disciplines
- Initially **analog** signals and systems implemented using resistors, capacitors, inductors, and transistors

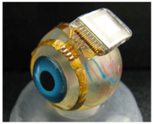


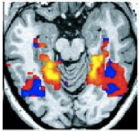
- Since the 1940s increasingly **digital** signals and systems implemented using computers and computer code (Matlab, Python, C, . . .)
  - Advantages of digital include stability and programmability
  - As computers have shrunk, digital signal processing has become ubiquitous

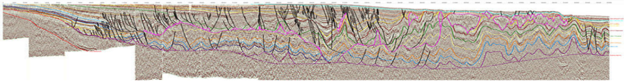
Digital Signal Processing Applications











Discrete Time Signals and Systems

- This edX course consists of one-half of the core Electrical and Computer Engineering course entitled "Signals and Systems" taught at Rice University in Houston, Texas, USA (see www.dsp.rice.edu)
- Goals: Develop intuition into and learn how to reason analytically about signal processing problems
- Video lectures, primary sources, supplemental materials, practice exercises, homework, programming case studies, final exam
- Integrated Matlab!
- The course comes in two halves
  - Part 1: Time Domain
  - Part 2: Frequency Domain

Course Outline

- Part 1: Time Domain
  - Week 1: Types of Signals
  - Week 2: Signals Are Vectors
  - Week 3: Systems
  - Week 4: Convolution
  - Week 5: Study Week, Practice Exam, Final Exam
- Part 2: Frequency Domain
  - Week 1: Discrete Fourier Transform (DFT)
  - Week 2: Discrete-Time Fourier Transform (DTFT)
  - Week 3: z Transform
  - Week 4: Analysis and Design of Discrete-Time Filters
  - Week 5: Study Week, Practice Exam, Final Exam

What You Should Do Each Week

- Watch the Lecture videos
- Do the Exercises (on the page to the right of the videos)
- As necessary, refer to the lesson's Supplemental Resources (the page to the right of the exercises)
- Do the homework problems
- Some weeks will also have graded MATLAB case study homework problems

Logistics and Grading

- How to get help: **Course Discussion** page
  - Use a thread set up for a particular topic, or
  - Start a new thread
- Rules for discussion
  - Be respectful and helpful
  - Do not reveal answers to any problem that will be graded
- Grading

Quick Questions	15%
Homework	30%
Homework Free Response Questions	15%
Pre-Exam Survey	5%
Final exam	30%
Post-Exam Survey	5%

- Passing grade: 60%

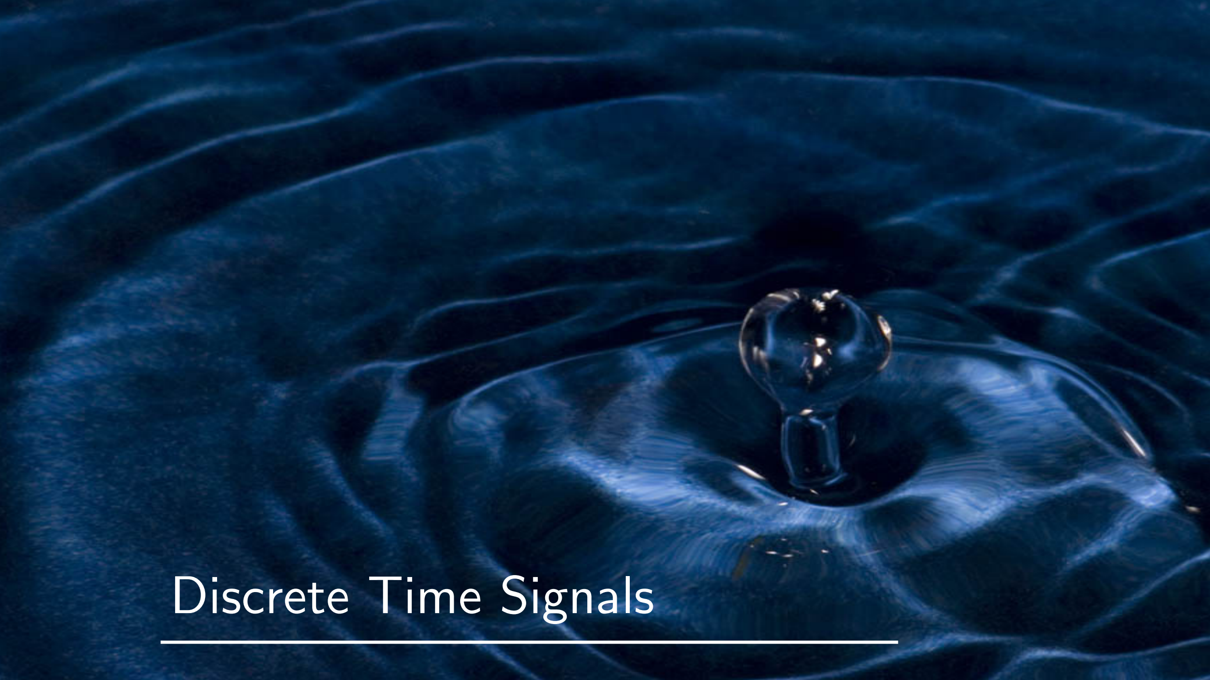
Supplemental Resources

- After the video lecture and a practice exercise or two, you will often see additional **Supplemental Resources**
- Sometimes these will contain background material to provide motivation for the topic
- Sometimes these will provide a refresher of pre-requisite concepts
- Sometimes these will provide deeper explanations of the content (more rigorous proofs, etc.)
- Sometimes a particular signal processing application will be showcased
- **Important:** Though the content in these resources will not be assessed in the homework or exam, you may find that they help you to understand a concept better or increase your interest in it

Before You Start

- **Important:** This is a mathematical treatment of signals and systems (no pain, no gain!)
- Please make sure you have a solid understanding of
  - Complex numbers and arithmetic
  - Linear algebra (vectors, matrices, dot products, eigenvectors, bases . . .)
  - Series (finite and infinite)
  - Calculus of a single variable (derivatives and integrals)
  - Matlab
- To test your readiness or refresh your knowledge, visit the "Pre-class Mathematics Refresher" section of the course

Discrete Time Signals and Systems



Discrete Time Signals

Signals

DEFINITION

Signal (n): A detectable physical quantity ... by which messages or information can be transmitted (Merriam-Webster)

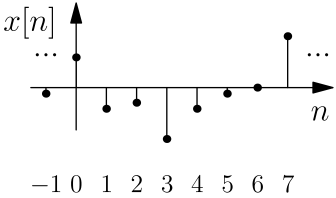
- Signals carry **information**
- Examples:
  - Speech signals transmit language via acoustic waves
  - Radar signals transmit the position and velocity of targets via electromagnetic waves
  - Electrophysiology signals transmit information about processes inside the body
  - Financial signals transmit information about events in the economy
- **Signal processing systems** manipulate the information carried by signals
- This is a course about signals and systems

Signals are Functions

DEFINITION

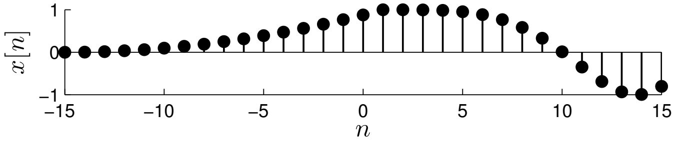
A **signal** is a function that maps an independent variable to a dependent variable.

- Signal $x[n]$: each value of n produces the value $x[n]$
- In this course, we will focus on **discrete-time** signals:
  - Independent variable is an **integer**: $n \in \mathbb{Z}$ (will refer to as time)
  - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or $\mathbb{C}$

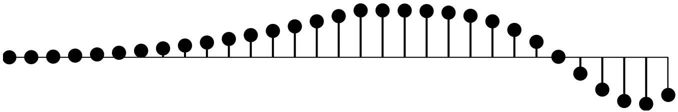


Plotting Real Signals

- When $x[n] \in \mathbb{R}$ (ex: temperature in a room at noon on Monday), we use one signal plot

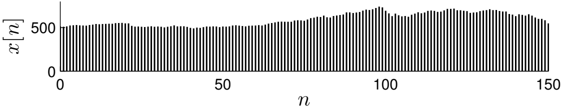


- When it is clear from context, we will often suppress the labels on one or both axes, like this

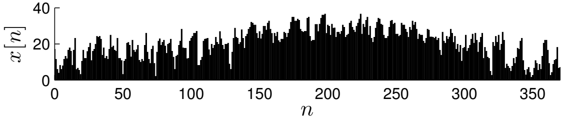


A Menagerie of Signals

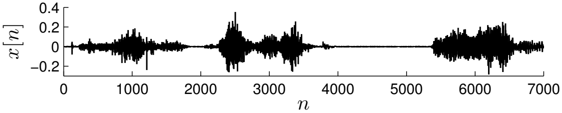
- Google Share daily share price for 5 months



- Temperature at Houston Intercontinental Airport in 2013 (Celcius)

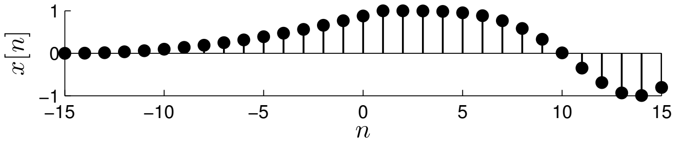


- Excerpt from Shakespeare's *Hamlet*

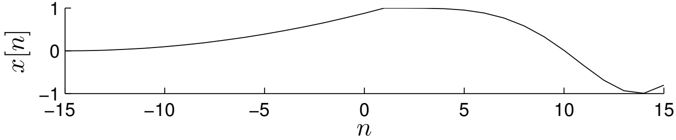


Plotting Signals Correctly

- In a discrete-time signal $x[n]$, the independent variable n is discrete (integer)
- To plot a discrete-time signal in a program like Matlab, you should use the `stem` or similar command and not the `plot` command
- Correct:



- Incorrect:

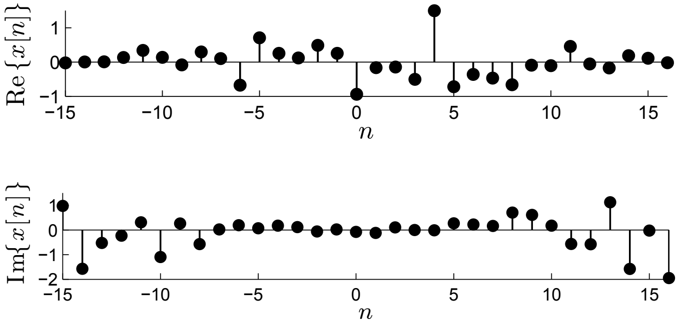


Plotting Complex Signals

- Recall that a complex number $a \in \mathbb{C}$ can be equivalently represented two ways:
  - Polar form: $a = |a|\, e^{j\angle a}$
  - Rectangular form: $a = \mathrm{Re}\{a\} + j\,\mathrm{Im}\{a\}$
- Here $j = \sqrt{-1}$ (engineering notation; mathematicians use $i = \sqrt{-1}$)
- When $x[n] \in \mathbb{C}$ (ex: magnitude and phase of an electromagnetic wave), we use <u>two</u> signal plots
  - Rectangular form: $x[n] = \mathrm{Re}\{x[n]\} + j\,\mathrm{Im}\{x[n]\}$
  - Polar form: $x[n] = |x[n]|\, e^{j\angle x[n]}$

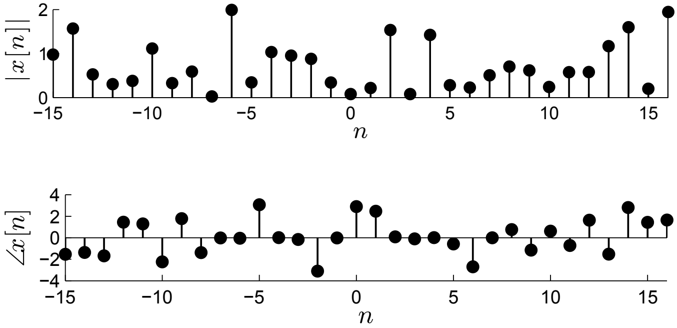
Plotting Complex Signals (Rectangular Form)

- Rectangular form: $x[n] = \mathrm{Re}\{x[n]\} + j\,\mathrm{Im}\{x[n]\} \in \mathbb{C}$



Plotting Complex Signals (Polar Form)

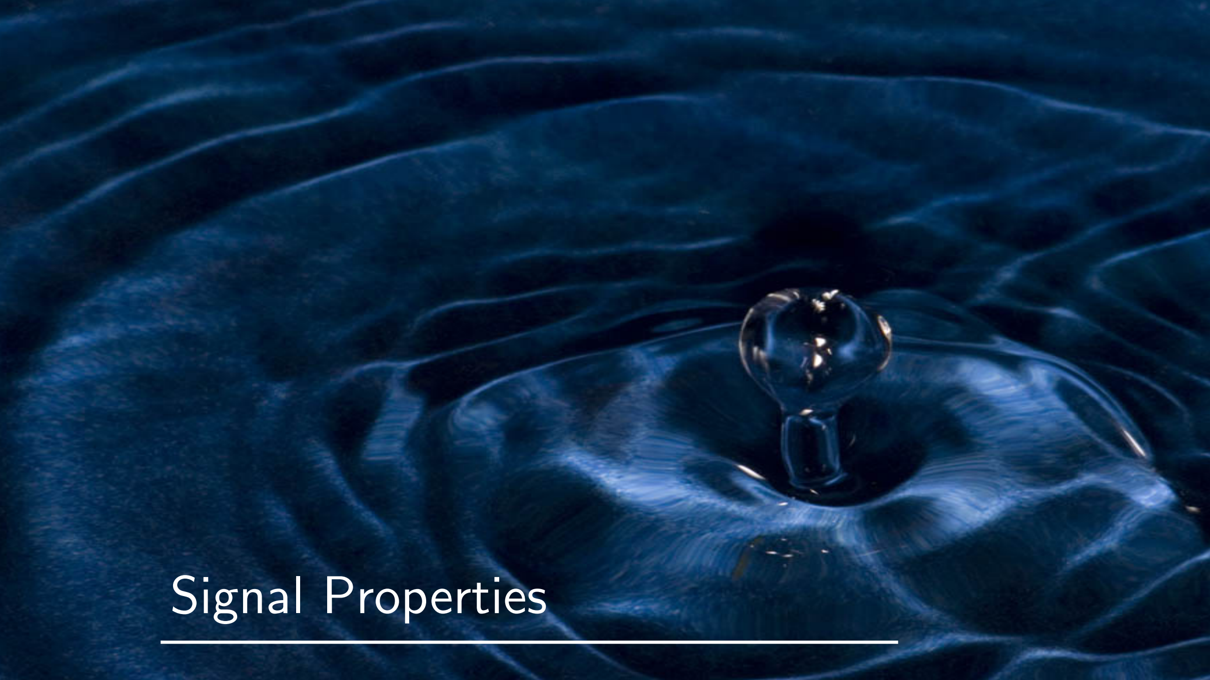
- Polar form: $x[n] = |x[n]|\, e^{j\angle(x[n])} \in \mathbb{C}$



Summary

- Discrete-time signals
  - Independent variable is an integer: $n \in \mathbb{Z}$ (will refer to as time)
  - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or $\mathbb{C}$

- Plot signals correctly!



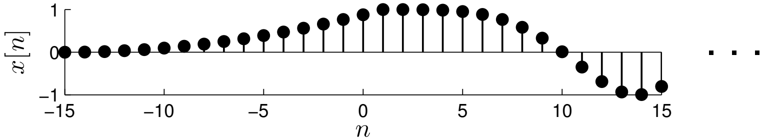
Signal Properties

Signal Properties

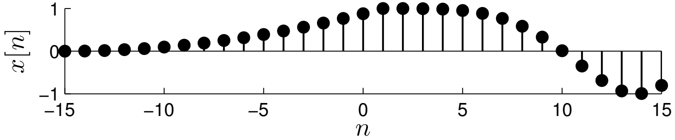
- Infinite/finite-length signals
- Periodic signals
- Causal signals
- Even/odd signals
- Digital signals

Finite/Infinite-Length Signals

- An **infinite-length** discrete-time signal $x[n]$ is defined for all $n \in \mathbb{Z}$, i.e., $-\infty < n < \infty$



- A **finite-length** discrete-time signal $x[n]$ is defined <u>only</u> for a finite range of $N_1 \leq n \leq N_2$



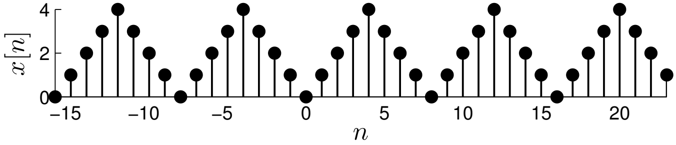
- Important: a finite-length signal is <u>undefined</u> for $n < N_1$ and $n > N_2$

Periodic Signals

DEFINITION

A discrete-time signal is **periodic** if it repeats with period $N \in \mathbb{Z}$:

$$x[n + mN] = x[n] \quad \forall\, m \in \mathbb{Z}$$



Notes:

- The period N must be an integer
- A periodic signal is infinite in length

DEFINITION

A discrete-time signal is **aperiodic** if it is not periodic

Converting between Finite and Infinite-Length Signals

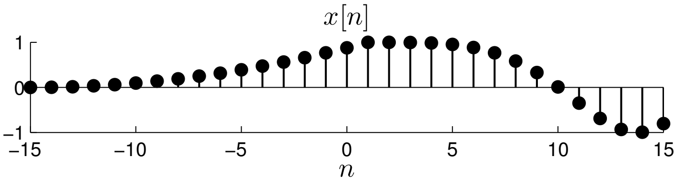
- Convert an infinite-length signal into a finite-length signal by **windowing**

- Convert a finite-length signal into an infinite-length signal by either
  - (infinite) **zero padding**, or
  - **periodization**

Windowing

- Converts a longer signal into a shorter one

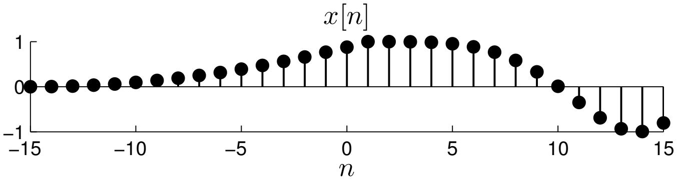
$$y[n] = \begin{cases} x[n] & N_1 \le n \le N_2 \\ 0 & \text{otherwise} \end{cases}$$



Zero Padding

- Converts a shorter signal into a longer one
- Say $x[n]$ is defined for $N_1 \le n \le N_2$
- Given $N_0 \le N_1 \le N_2 \le N_3$

$$y[n] = \begin{cases} 0 & N_0 \le n < N_1 \\ x[n] & N_1 \le n \le N_2 \\ 0 & N_2 < n \le N_3 \end{cases}$$

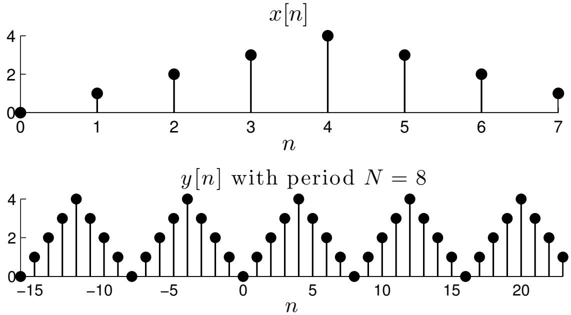


Periodization

- Converts a finite-length signal into an infinite-length, periodic signal

- Given finite-length $x[n]$, replicate $x[n]$ periodically with period N

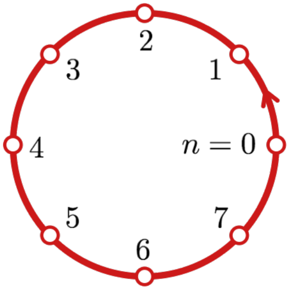
$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[n - mN], \quad n \in \mathbb{Z} \\
&= \cdots + x[n+2N] + x[n+N] + x[n] + x[n-N] + x[n-2N] + \cdots
\end{aligned}
$$



Useful Aside – Modular Arithmetic

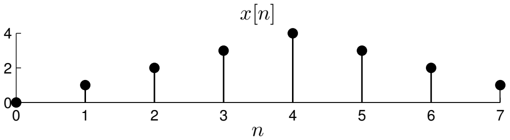
- Modular arithmetic with **modulus** N (mod-N) takes place on a **clock** with N "hours"
  - Ex: $(12)_8$ ("twelve mod eight")

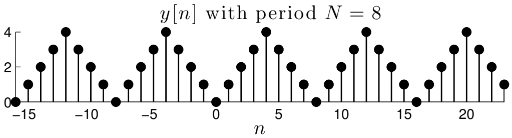
- Modulo arithmetic is inherently **periodic**
  - Ex: $\ldots\ (-12)_8 \ = \ (-4)_8 \ = \ (4)_8 \ = \ (12)_8 \ = \ (20)_8 \ \ldots$



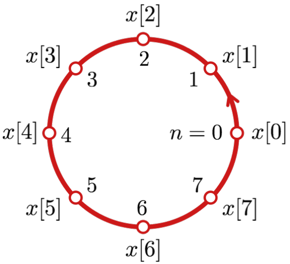
Periodization via Modular Arithmetic

- Consider a length-N signal $x[n]$ defined for $0 \leq n \leq N-1$
- A convenient way to express periodization with period N is $\quad y[n] = x[(n)_N], \quad n \in \mathbb{Z}$

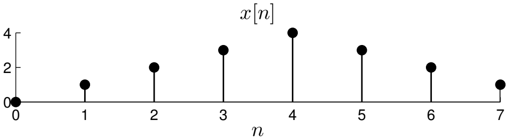


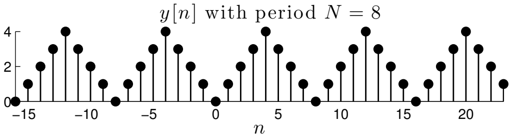


- Important interpretation
  - Infinite-length signals live on the (infinite) number **line**
  - Periodic signals live on a **circle** – a clock with N “hours”



Finite-Length and Periodic Signals are Equivalent



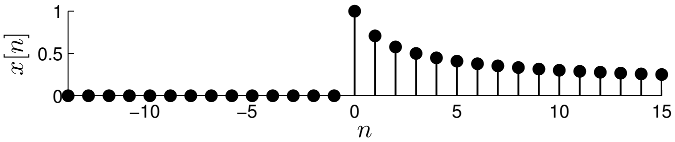


- All of the information in a periodic signal is contained in **one period** (of finite length)
- Any finite-length signal can be periodized
- Conclusion: We can and will think of finite-length signals and periodic signals interchangeably
- We can choose the most convenient viewpoint for solving any given problem
  - Application: Shifting finite length signals

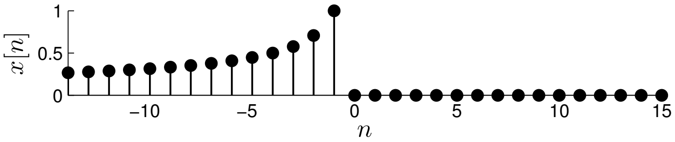
Causal Signals

DEFINITION

A signal $x[n]$ is **causal** if $x[n] = 0$ for all $n < 0$.



- A signal $x[n]$ is **anti-causal** if $x[n] = 0$ for all $n \geq 0$

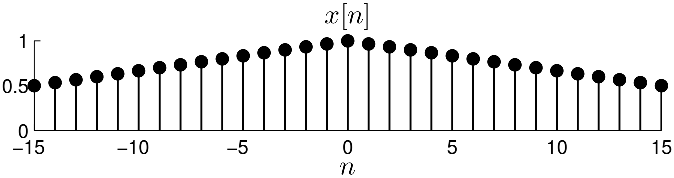


- A signal $x[n]$ is **acausal** if it is not causal

Even Signals

DEFINITION

A real signal $x[n]$ is **even** if $x[-n] = x[n]$

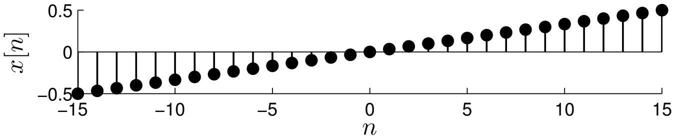


- Even signals are symmetrical around the point $n = 0$

Odd Signals

DEFINITION

A real signal $x[n]$ is **odd** if $x[-n] = -x[n]$



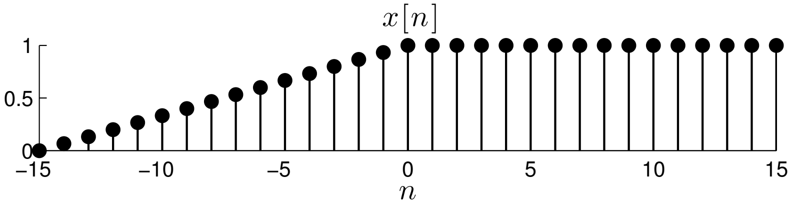
- Odd signals are anti-symmetrical around the point $n = 0$

Even+Odd Signal Decomposition

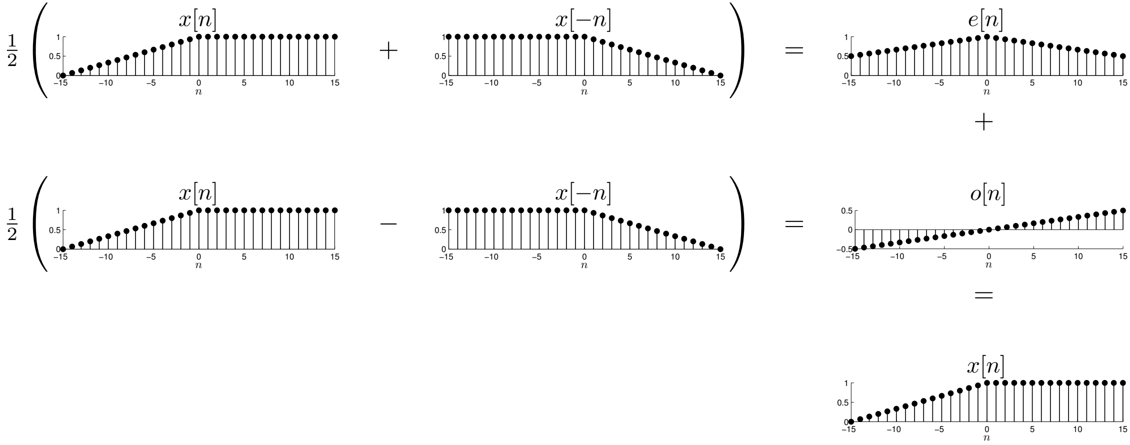
- **Useful fact:** Every signal $x[n]$ can be decomposed into the sum of its even part + its odd part
- Even part: $e[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$ (easy to verify that $e[n]$ is even)
- Odd part: $o[n] = \frac{1}{2}\left(x[n] - x[-n]\right)$ (easy to verify that $o[n]$ is odd)
- **Decomposition** $x[n] = e[n] + o[n]$
- Verify the decomposition:

$$\begin{aligned} e[n] + o[n] &= \frac{1}{2}(x[n] + x[-n]) + \frac{1}{2}(x[n] - x[-n]) \\ &= \frac{1}{2}(x[n] + x[-n] + x[n] - x[-n]) \\ &= \frac{1}{2}(2\,x[n]) = x[n] \quad \checkmark \end{aligned}$$

Even+Odd Signal Decomposition in Pictures

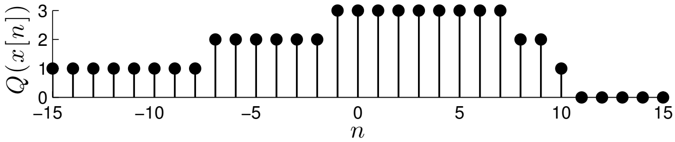


Even+Odd Signal Decomposition in Pictures



Digital Signals

- **Digital signals** are a special sub-class of discrete-time signals
  - Independent variable is still an integer: $n \in \mathbb{Z}$
  - Dependent variable is from a **finite set of integers**: $x[n] \in \{0, 1, \ldots, D-1\}$
  - Typically, choose $D = 2^q$ and represent each possible level of $x[n]$ as a digital code with q bits
  - Ex: Digital signal with $q = 2$ bits $\Rightarrow D = 2^2 = 4$ levels



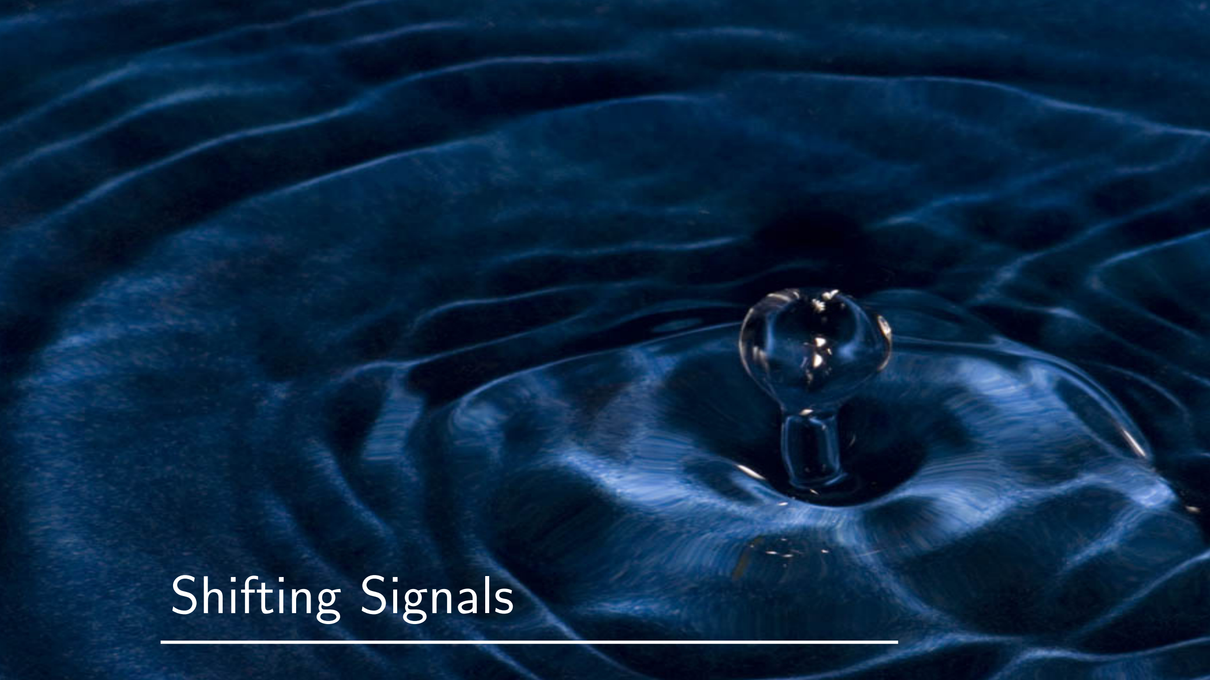
  - Ex: Compact discs use $q = 16$ bits $\Rightarrow D = 2^{16} = 65536$ levels

Summary

- Signals can be classified many different ways (real/complex, infinite/finite-length, periodic/aperiodic, causal/acausal, even/odd, ...)

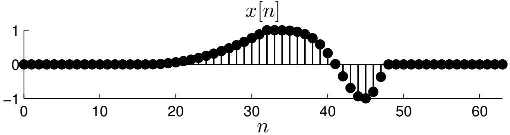
- Finite-length signals are equivalent to periodic signals; modulo arithmetic useful

Shifting Signals

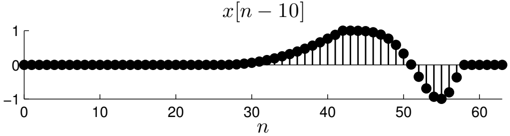


Shifting Infinite-Length Signals

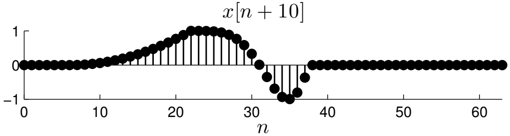
- Given an infinite-length signal $x[n]$, we can **shift** it back and forth in time via $x[n-m]$, $m \in \mathbb{Z}$



- When $m > 0$, $x[n-m]$ shifts to the **right** (forward in time, delay)



- When $m < 0$, $x[n-m]$ shifts to the **left** (back in time, advance)

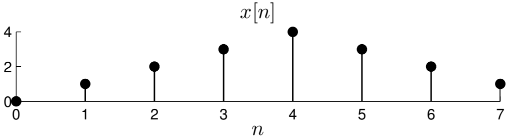


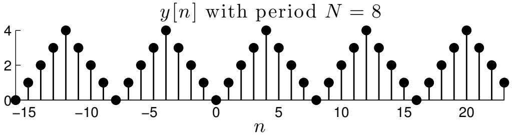
Periodic Signals and Modular Arithmetic

- A convenient way to express a signal $y[n]$ that is periodic with period N is

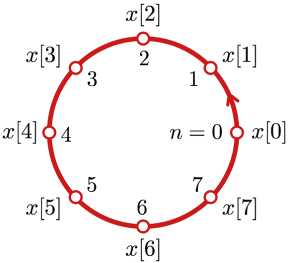
$$y[n] = x[(n)_N], \quad n \in \mathbb{Z}$$

where $x[n]$, defined for $0 \leq n \leq N-1$, comprises one period



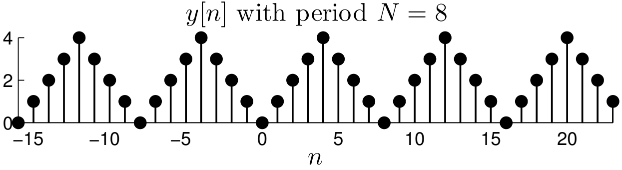


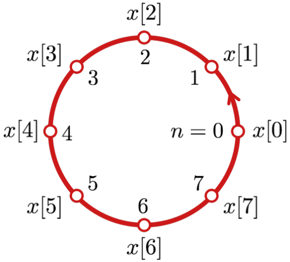
- Important interpretation
  - Infinite-length signals live on the (infinite) number line
  - Periodic signals live on a circle – a clock with N "hours"



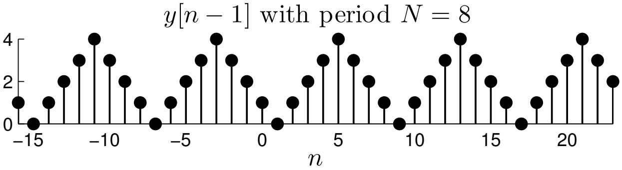
Shifting Periodic Signals

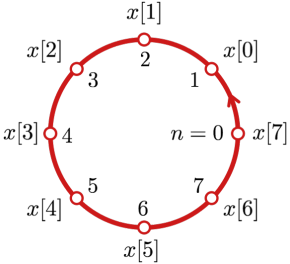
- Periodic signals can also be shifted; consider $y[n] = x[(n)_N]$





- Shift one sample into the future: $y[n-1] = x[(n-1)_N]$





Shifting Finite-Length Signals

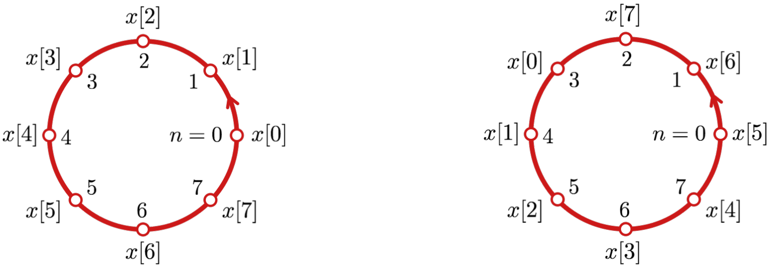
- Consider finite-length signals x and v defined for $0 \leq n \leq N-1$ and suppose "$v[n] = x[n-1]$"

$$\begin{aligned} v[0] &= ?? \\ v[1] &= x[0] \\ v[2] &= x[1] \\ v[3] &= x[2] \\ &\vdots \\ v[N-1] &= x[N-2] \\ ?? &= x[N-1] \end{aligned}$$

- What to put in $v[0]$? What to do with $x[N-1]$? We don't want to invent/lose information
- Elegant solution: Assume x and v are both periodic with period N; then $v[n] = x[(n-1)_N]$
- This is called a **periodic** or **circular shift** (see `circshift` and `mod` in Matlab)

Circular Shift Example

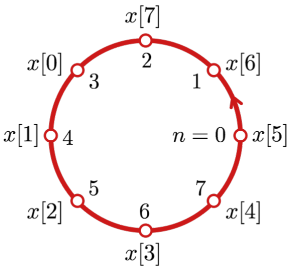
- Elegant formula for circular shift of $x[n]$ by m time steps: $x[(n-m)_N]$
- Ex: x and v defined for $0 \le n \le 7$, that is, $N = 8$. Find $v[n] = x[(n-3)_8]$



Circular Shift Example

- Elegant formula for circular shift of $x[n]$ by m time steps: $x[(n-m)_N]$
- Ex: x and v defined for $0 \leq n \leq 7$, that is, $N = 8$. Find $v[n] = x[(n-m)_N]$

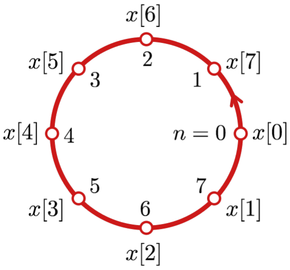
$$
\begin{aligned}
v[0] &= x[5] \\
v[1] &= x[6] \\
v[2] &= x[7] \\
v[3] &= x[0] \\
v[4] &= x[1] \\
v[5] &= x[2] \\
v[6] &= x[3] \\
v[7] &= x[4]
\end{aligned}
$$



Circular Time Reversal

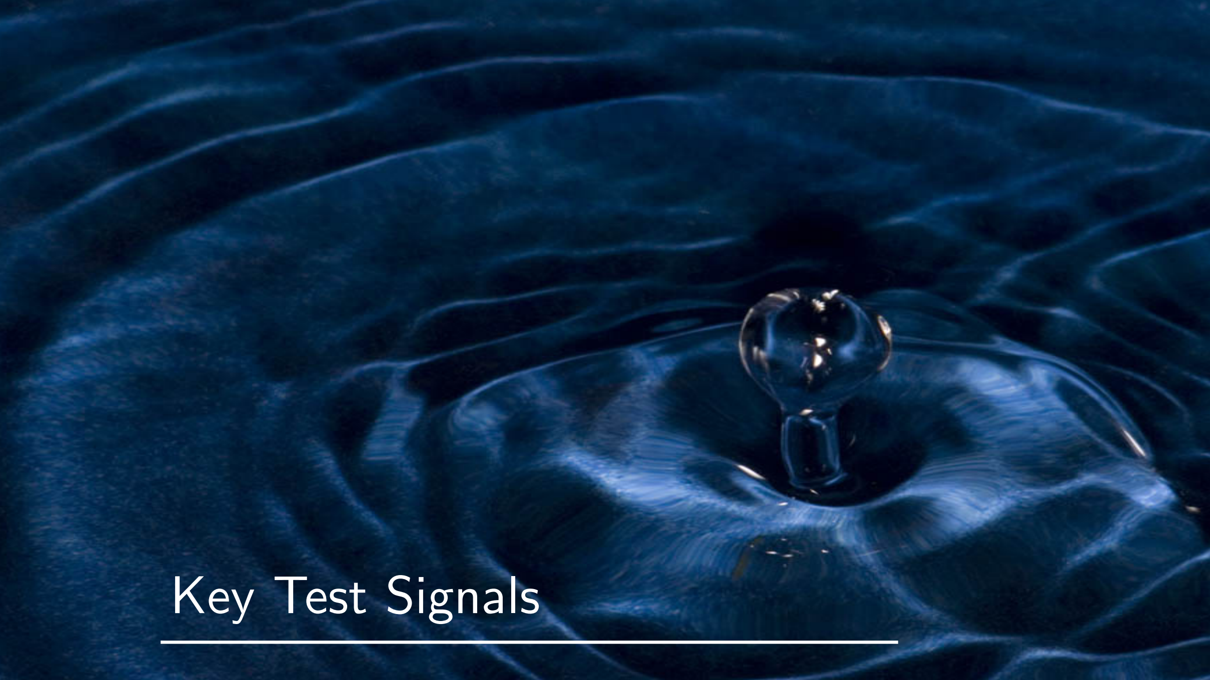
- For infinite length signals, the transformation of reversing the time axis $x[-n]$ is obvious
- Not so obvious for periodic/finite-length signals
- Elegant formula for reversing the time axis of a periodic/finite-length signal: $x[(-n)_N]$
- Ex: x and v defined for $0 \leq n \leq 7$, that is, $N = 8$. Find $v[n] = x[(-n)_N]$

$$
\begin{aligned}
v[0] &= x[0] \\
v[1] &= x[7] \\
v[2] &= x[6] \\
v[3] &= x[5] \\
v[4] &= x[4] \\
v[5] &= x[3] \\
v[6] &= x[2] \\
v[7] &= x[1]
\end{aligned}
$$



Summary

- Shifting a signal moves it forward or backward in time
- Modulo arithmetic provides and easy way to shift periodic signals



Key Test Signals

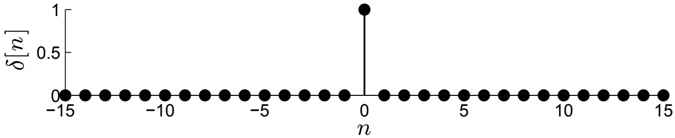
A Toolbox of Test Signals

- Delta function
- Unit step
- Unit pulse
- Real exponential
- Still to come: sinusoids, complex exponentials
- **Note:** We will introduce the test signals as infinite-length signals, but each has a finite-length equivalent

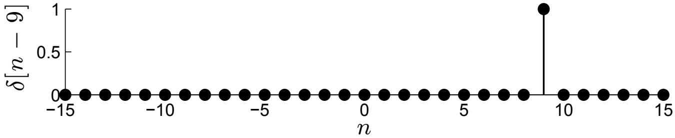
Delta Function

DEFINITION

The **delta function** (aka unit impulse) $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$



- The shifted delta function $\delta[n-m]$ peaks up at $n = m$; here $m = 9$

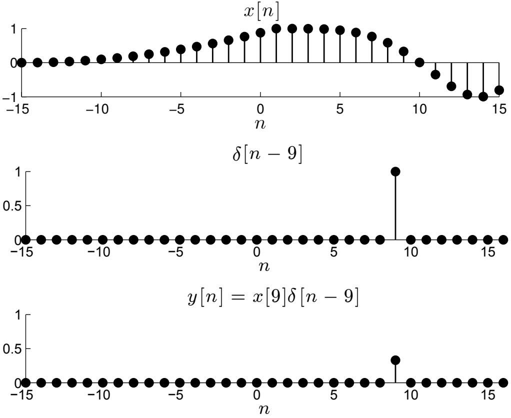


Delta Functions Sample

- Multiplying a signal by a shifted delta function picks out one sample of the signal and sets all other samples to zero

$$y[n] \; = \; x[n]\,\delta[n-m] \; = \; x[m]\,\delta[n-m]$$

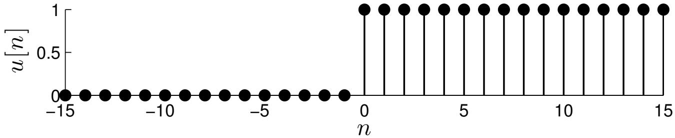
- Important: m is a fixed constant, and so $x[m]$ is a constant (and not a signal)



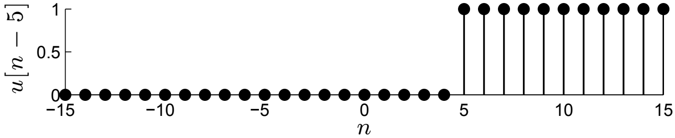
Unit Step

DEFINITION

The **unit step** $$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



- The shifted unit step $u[n-m]$ jumps from 0 to 1 at $n = m$; here $m = 5$

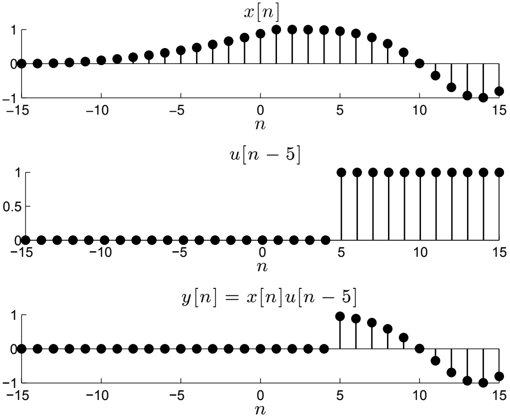


Unit Step Selects Part of a Signal

- Multiplying a signal by a shifted unit step function zeros out its entries for $n < m$

$$y[n] \;=\; x[n]\,u[n-m]$$

(Note: For $m = 0$, this makes $y[n]$ causal)

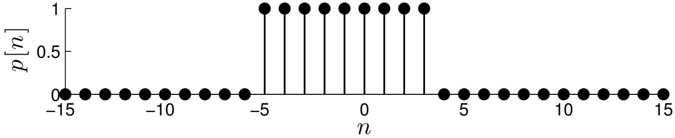


Unit Pulse

DEFINITION

The **unit pulse** (aka boxcar) $$p[n] = \begin{cases} 0 & n < N_1 \\ 1 & N_1 \le n \le N_2 \\ 0 & n > N_2 \end{cases}$$

- Ex: $p[n]$ for $N_1 = -5$ and $N_2 = 3$



- One of many different formulas for the unit pulse

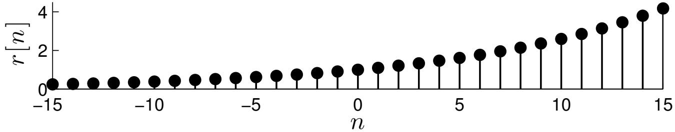
$$p[n] = u[n - N_1] - u[n - (N_2 + 1)]$$

Real Exponential

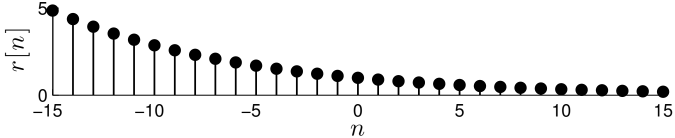
DEFINITION

The **real exponential** $r[n] = a^n$, $a \in \mathbb{R}$, $a \geq 0$

- For $a > 1$, $r[n]$ shrinks to the left and grows to the right; here $a = 1.1$



- For $0 < a < 1$, $r[n]$ grows to the left and shrinks to the right; here $a = 0.9$



Summary

- We will use our test signals often, especially the delta function and unit step

Sinusoids

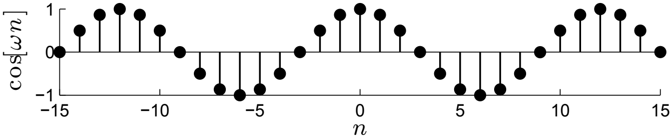
Sinusoids

- Sinusoids appear in myriad disciplines, in particular signal processing
- They are the basis (literally) of Fourier analysis (DFT, DTFT)
- We will introduce
    - Real-valued sinusoids
    - (Complex) sinusoid
    - Complex exponential

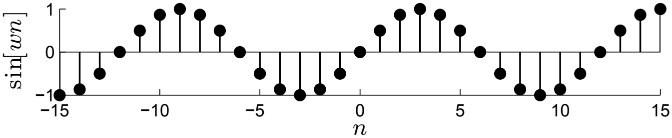
Sinusoids

- There are two natural real-valued sinusoids: $\cos(\omega n + \phi)$ and $\sin(\omega n + \phi)$
- **Frequency:** ω (units: radians/sample)
- **Phase:** ϕ (units: radians)

- $\cos(\omega n)$ (even)

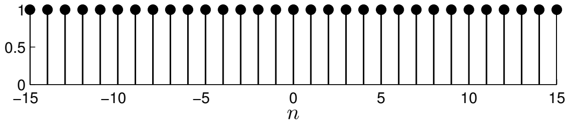


- $\sin(\omega n)$ (odd)

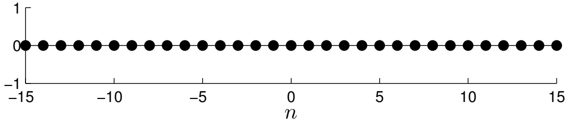


Sinusoid Examples

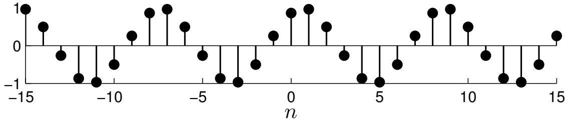
- $\cos(0n)$



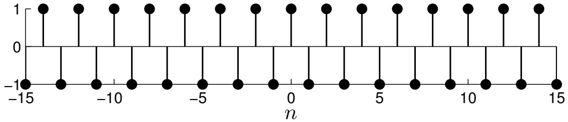
- $\sin(0n)$



- $\sin(\frac{\pi}{4}n + \frac{2\pi}{6})$



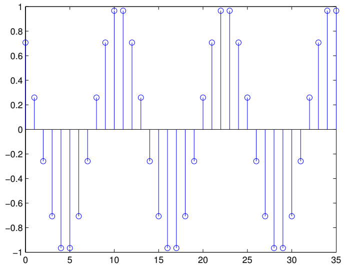
- $\cos(\pi n)$



Get Comfortable with Sinusoids!

- It's easy to play around in Matlab to get comfortable with the properties of sinusoids

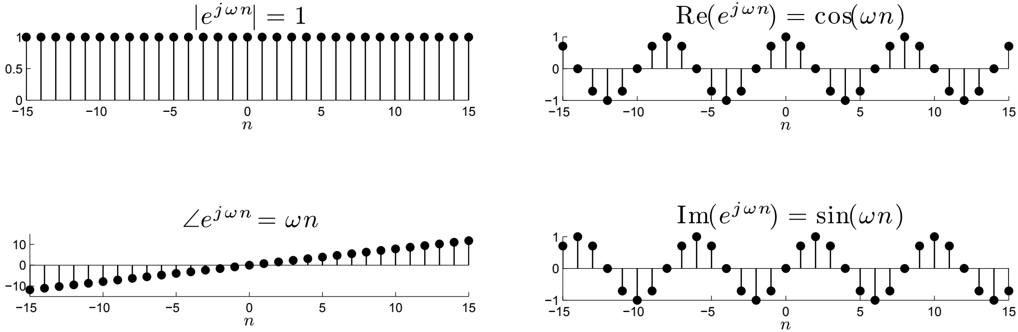
```
N=36;
n=0:N-1;
omega=pi/6;
phi=pi/4;
x=cos(omega*n+phi);
stem(n,x)
```



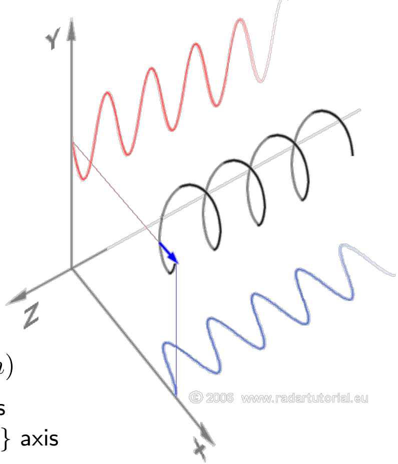
Complex Sinusoid

- The complex-valued sinusoid combines both the $\cos$ and $\sin$ terms (via Euler's identity)

$$e^{j(\omega n+\phi)} = \cos(\omega n+\phi) + j\sin(\omega n+\phi)$$



A Complex Sinusoid is a Helix



$$e^{j(\omega n+\phi)} = \cos(\omega n+\phi) + j\sin(\omega n+\phi)$$

- A complex sinusoid is a **helix** in 3D space $(\mathrm{Re}\{\}, \mathrm{Im}\{\}, n)$
  - Real part $(\cos$ term$)$ is the projection onto the $\mathrm{Re}\{\}$ axis
  - Imaginary part $(\sin$ term$)$ is the projection onto the $\mathrm{Im}\{\}$ axis
- Frequency ω determines rotation speed and direction of helix
  - $\omega > 0 \Rightarrow$ anticlockwise rotation
  - $\omega < 0 \Rightarrow$ clockwise rotation

Complex Sinusoid is a Helix (Animation)

- **Complex sinusoid animation**

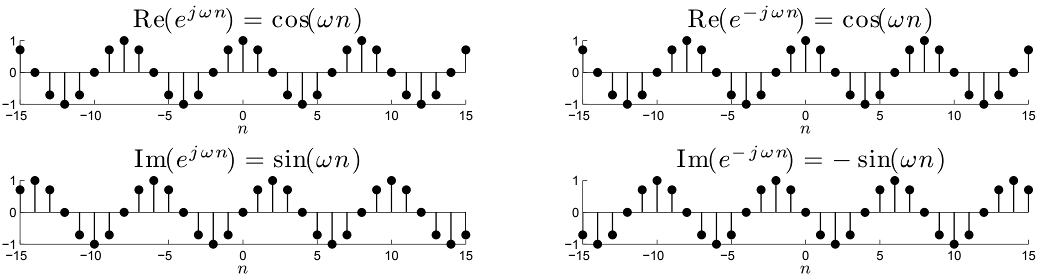
Negative Frequency

- Negative frequency is nothing to be afraid of! Consider a sinusoid with a negative frequency $-\omega$

$$e^{j(-\omega)n} \;=\; e^{-j\omega n} \;=\; \cos(-\omega n) + j\sin(-\omega n) \;=\; \cos(\omega n) - j\sin(\omega n)$$

- Also note: $e^{j(-\omega)n} \;=\; e^{-j\omega n} \;=\; \left(e^{j\omega n}\right)^*$

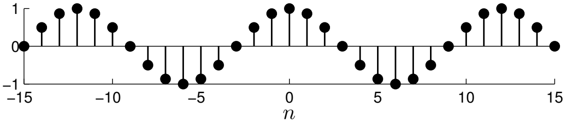
- Bottom line: negating the frequency is equivalent to complex conjugating a complex sinusoid, which flips the sign of the imaginary, $\sin$ term



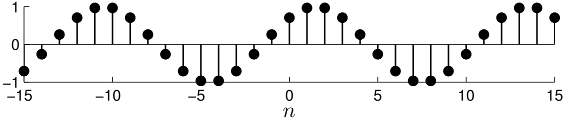
Phase of a Sinusoid $e^{j(\omega n+\phi)}$

- ϕ is a (frequency independent) shift that is referenced to one period of oscillation

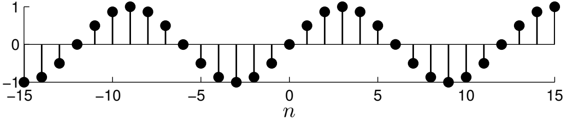
- $\cos\left(\frac{\pi}{6}n - 0\right)$



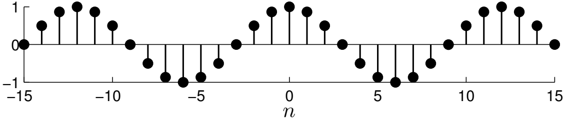
- $\cos\left(\frac{\pi}{6}n - \frac{\pi}{4}\right)$



- $\cos\left(\frac{\pi}{6}n - \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{6}n\right)$



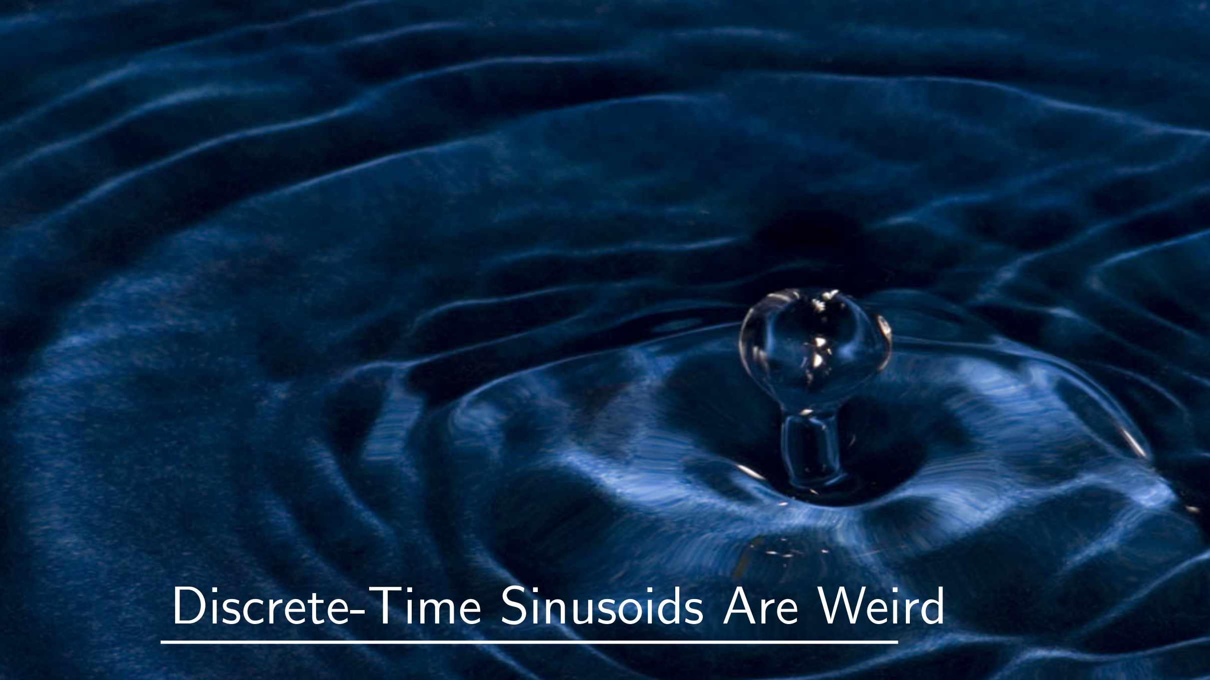
- $\cos\left(\frac{\pi}{6}n - 2\pi\right) = \cos\left(\frac{\pi}{6}n\right)$



Summary

- Sinusoids play a starring role in both the theory and applications of signals and systems
- A sinusoid has a **frequency** and a **phase**
- A complex sinusoid is a helix in three-dimensional space and naturally induces the sine and cosine
- Negative frequency is nothing to be scared by; it just means that the helix spins backwards

Discrete-Time Sinusoids Are Weird



Discrete-Time Sinusoids are Weird!

- Discrete-time sinusoids $e^{j(\omega n+\phi)}$ have two counterintuitive properties
- Both involve the frequency ω
- Weird property $\#1$: <u>Aliasing</u>
- Weird property $\#2$: <u>Aperiodicity</u>

Weird Property #1: Aliasing of Sinusoids

- Consider two sinusoids with two different frequencies
  - $\omega \quad \Rightarrow \quad x_1[n] = e^{j(\omega n+\phi)}$
  - $\omega + 2\pi \quad \Rightarrow \quad x_2[n] = e^{j((\omega+2\pi)n+\phi)}$

- But note that

$$x_2[n] = e^{j((\omega+2\pi)n+\phi)} = e^{j(\omega n+\phi)+j2\pi n} = e^{j(\omega n+\phi)}\ e^{j2\pi n} = e^{j(\omega n+\phi)} = x_1[n]$$

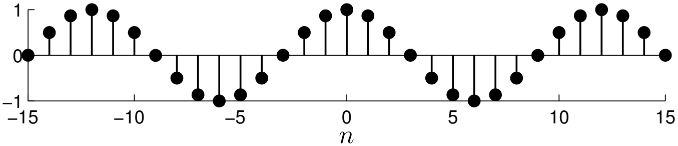
- The signals x_1 and x_2 have different frequencies but are **identical**!

- We say that x_1 and x_2 are aliases; this phenomenon is called **aliasing**

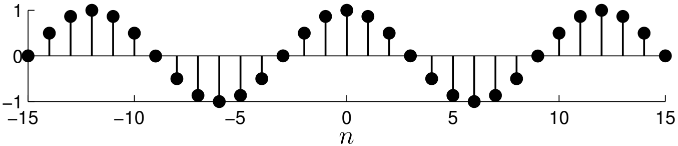
- Note: Any integer multiple of 2π will do; try with $x_3[n] = e^{j((\omega+2\pi m)n+\phi)}$, $m \in \mathbb{Z}$

Aliasing of Sinusoids – Example

- $x_1[n] = \cos\left(\frac{\pi}{6}n\right)$



- $x_2[n] = \cos\left(\frac{13\pi}{6}n\right) = \cos\left(\left(\frac{\pi}{6} + 2\pi\right)n\right)$



Alias-Free Frequencies

- Since

$$x_3[n] = e^{j(\omega+2\pi m)n+\phi} = e^{j(\omega n+\phi)} = x_1[n] \quad \forall m \in \mathbb{Z}$$

  the only frequencies that lead to unique (distinct) sinusoids lie in an <u>interval of length 2π</u>

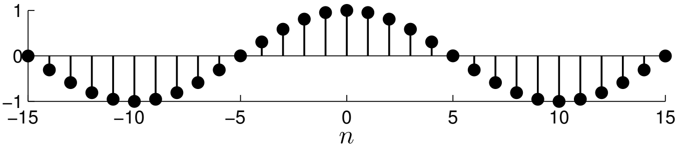
- Convenient to interpret the frequency ω as an **angle**
  (then aliasing is handled automatically; more on this later)

- Two intervals are typically used in the signal processing literature (and in this course)
  - $0 \leq \omega < 2\pi$
  - $-\pi < \omega \leq \pi$

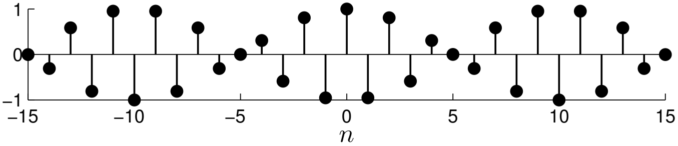
Low and High Frequencies

$$e^{j(\omega n+\phi)}$$

- **Low frequencies:** ω close to 0 or 2π rad
  Ex: $\cos\left(\frac{\pi}{10}n\right)$



- **High frequencies:** ω close to π or $-\pi$ rad
  Ex: $\cos\left(\frac{9\pi}{10}n\right)$



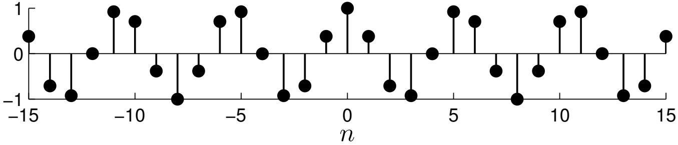
Weird Property #2: Periodicity of Sinusoids

- Consider $x_1[n] = e^{j(\omega n+\phi)}$ with frequency $\omega = \frac{2\pi k}{N}$, $\quad k, N \in \mathbb{Z}$ (harmonic frequency)

- It is easy to show that x_1 is periodic with period N, since

$$x_1[n+N] = e^{j(\omega(n+N)+\phi)} = e^{j(\omega n+\omega N+\phi)} = e^{j(\omega n+\phi)}\, e^{j(\omega N)} = e^{j(\omega n+\phi)}\, e^{j(\frac{2\pi k}{N} N)} = x_1[n] \checkmark$$

- Ex: $x_1[n] = \cos(\frac{2\pi 3}{16} n)$, $\;N = 16$



- Note: x_1 is periodic with the (smaller) period of $\frac{N}{k}$ when $\frac{N}{k}$ is an integer

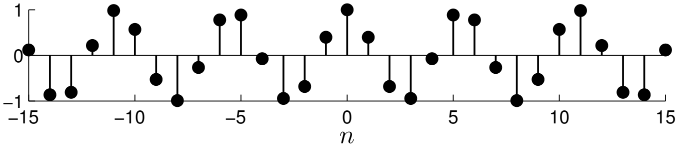
Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n+\phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$ (not harmonic frequency)

- Is x_2 periodic?

$$x_2[n+N] = e^{j(\omega(n+N)+\phi)} = e^{j(\omega n+\omega N+\phi)} = e^{j(\omega n+\phi)}\, e^{j(\omega N)} \neq x_1[n] \quad \text{NO!}$$

- Ex: $x_2[n] = \cos(1.16\, n)$



- If its frequency ω is not harmonic, then a sinusoid oscillates but is not periodic!

Harmonic Sinusoids

$$e^{j(\omega n+\phi)}$$

- Semi-amazing fact: The **only** periodic discrete-time sinusoids are those with **harmonic frequencies**

$$\omega = \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$$

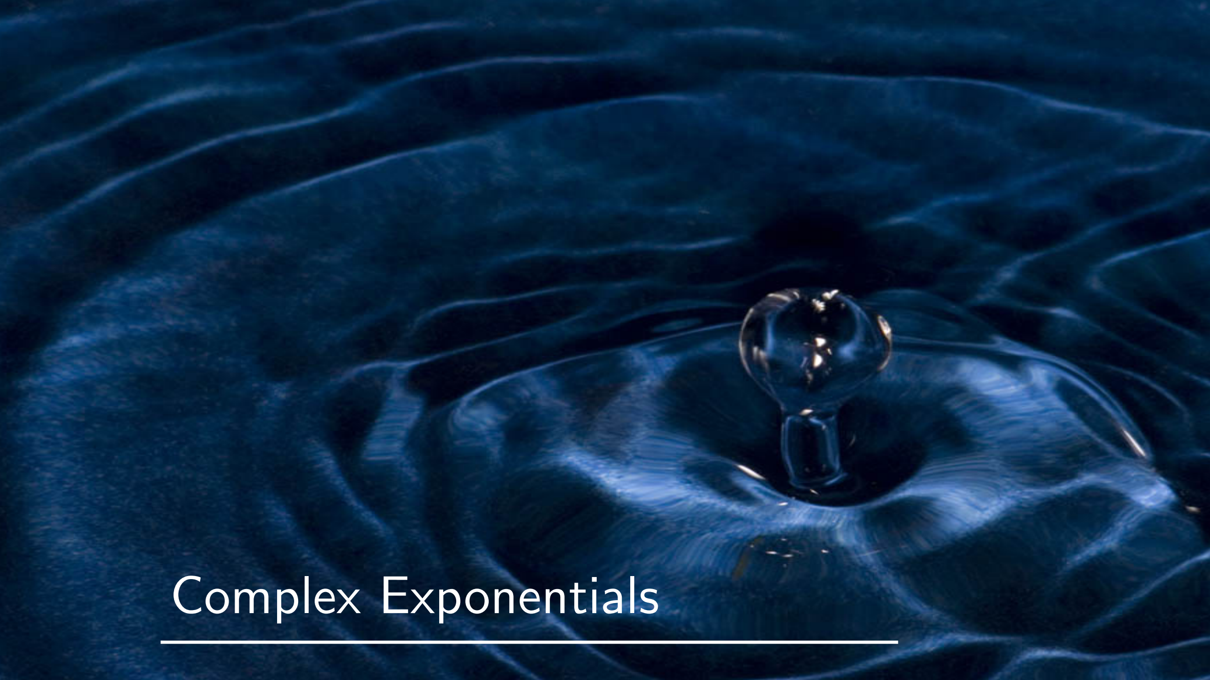
- Which means that
  - **Most** discrete-time sinusoids are **not** periodic!
  - The harmonic sinusoids are somehow magical (they play a starring role later in the DFT)

Harmonic Sinusoids (Matlab)

- **Click here** to view a MATLAB demo that visualizes harmonic sinusoids.

Summary

- Discrete-time sinusoids $e^{j(\omega n+\phi)}$ have two counterintuitive properties
- Both involve the frequency ω
- Weird property #1: Aliasing
- Weird property #2: Aperidiocity
- The only sinusoids that are periodic: Harmonic sinusoids $e^{j(\frac{2\pi k}{N}n+\phi)}$, $n, k, N \in \mathbb{Z}$



Complex Exponentials

Complex Exponential

- Complex sinusoid $e^{j(\omega n+\phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$

- Generalize to $e^{\text{General Complex Numbers}}$

- Consider the general complex number $z = |z|\, e^{j\omega}$, $z \in \mathbb{C}$
  - $|z|$ = magnitude of z
  - $\omega = \angle(z)$, phase angle of z
  - Can visualize $z \in \mathbb{C}$ as a **point** in the **complex plane**

- Now we have

$$z^n = (|z|e^{j\omega})^n = |z|^n (e^{j\omega})^n = |z|^n e^{j\omega n}$$

  - $|z|^n$ is a **real exponential** $(a^n$ with $a = |z|)$
  - $e^{j\omega n}$ is a **complex sinusoid**

Complex Exponential is a Spiral

$$z^n \;=\; \left(|z|\, e^{j\omega}\right)^n \;=\; |z|^n\, e^{j\omega n}$$

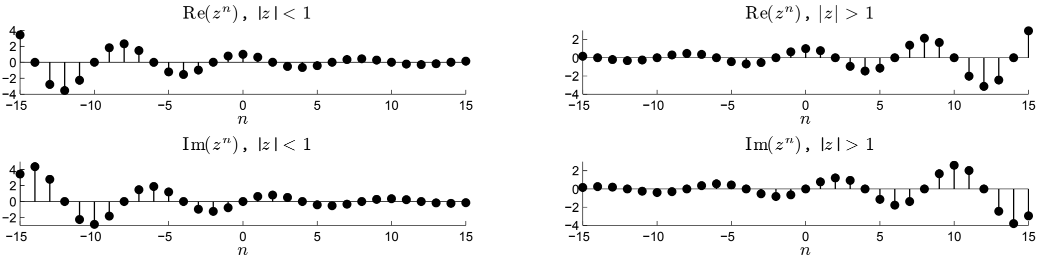
- $|z|^n$ is a **real exponential** envelope $(a^n$ with $a = |z|)$
- $e^{j\omega n}$ is a **complex sinusoid**
- z^n is a helix with expanding radius (spiral)

Complex Exponential is a Spiral

$$z^n \;=\; \left(|z|\, e^{j\omega n}\right)^n \;=\; |z|^n\, e^{j\omega n}$$

- $|z|^n$ is a **real exponential** envelope $\left(a^n \text{ with } a = |z|\right)$
- $e^{j\omega n}$ is a **complex sinusoid**

$|z| < 1$ $\qquad\qquad$ $|z| > 1$



Complex Exponentials and z Plane (Matlab)

- **Click here** to view a MATLAB demo plotting the signals z^n.

Summary

- Complex sinusoid $e^{j(\omega n+\phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$
- Complex exponential: Generalize $e^{j(\omega n+\phi)}$ to $e^{\text{General Complex Numbers}}$
- A complex exponential is the product of a real exponential and a complex sinusoid
- A complex exponential is a spiral in three-dimensional space