Data Structures and Algorithms (12)

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Higher Education Press, 2008.6 (the "Eleventh Five-Year" national planning textbook)

https://courses.edx.org/courses/PekingX/04830050x/2T2014/

Chapter 12 Advanced Data Structure

- 12.1 Multidimensional array
  - 12.1.1 Basic Concepts
  - 12.1.2 Structure of Array
  - 12.1.3 Storage of Array
  - 12.1.4 Declaration of Array
  - 12.1.5 Special Matrices Implemented by Arrays
  - 12.1.6 Sparse Matrix
- 12.2 Generalized List
- 12.3 Storage management
- 12.4 Trie
- 12.5 Improved BST

Basic Concepts

- Array is an ordered sequence with fixed number of elements and type.
- The size and type of static array must be specified at compile time
- Dynamic array is allocated memory at runtime

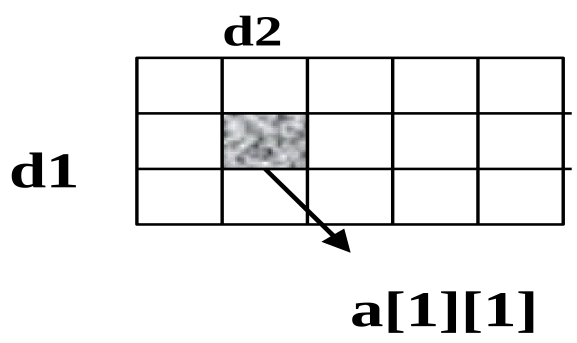
Basic Concepts

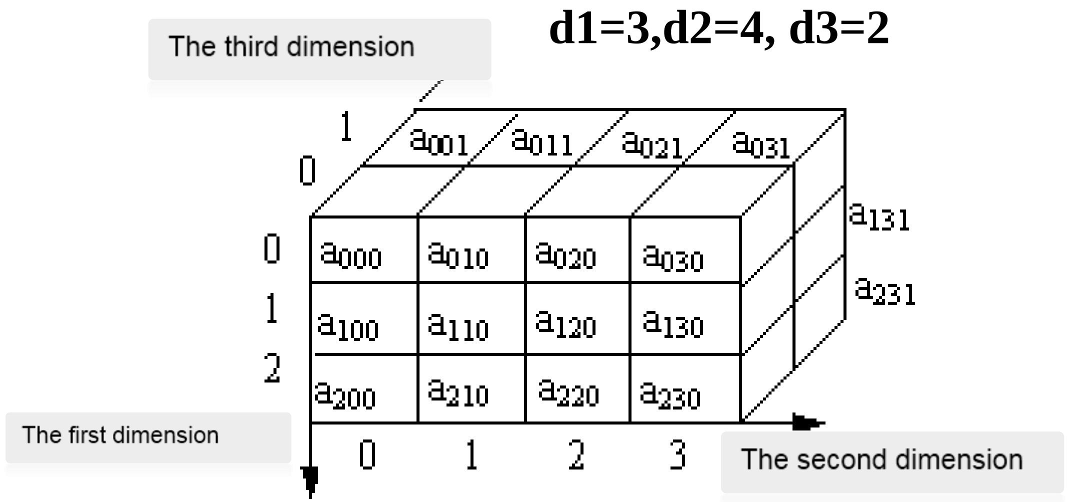
- Multidimensional array is an extension of one-dimensional array (vector).
- Vector of vectors make up an multidimensional array.
- Represented as

  ELEM A[c_1..d_1][c_2..d_2]…[c_n..d_n]
- c_i and d_i are upper and lower bounds of the indices in the i-th dimension. Thus, the total number of elements is: $\prod_{i=1}^{n}(d_i - c_i + 1)$

Structure of Array

d1=3,d2=5





2-dimensional array

3-dimensional array

d1[0..2], d2[0..3], d3[0..1] are the three dimensions respectively

Storage of Array

- Memory is one-dimensional, so arrays are stored linearly
  - Stored row by row (row-major)
  - Stored column by column (column-major)

$$X = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Row-Major in Pascal

a[1..k,1..m,1..n]

a_{111} a_{112} a_{113} … a_{11n}	a_{11*}
a_{121} a_{122} a_{123} … a_{12n}	a_{12*}
…………………………	
a_{1m1} a_{1m2} a_{1m3} … a_{1mn}	a_{1m*}
a_{211} a_{212} a_{213} … a_{21n}	a_{21*}
a_{221} a_{222} a_{223} … a_{22n}	a_{22*}
…………………………	
a_{2m1} a_{2m2} a_{2m3} … a_{2mn}	a_{2m*}

⋮

a_{k11} a_{k12} a_{k13} … a_{k1n}
a_{k21} a_{k22} a_{k23} … a_{k2n}
…………………………
a_{km1} a_{km2} a_{km3} … a_{kmn}

Column-Major in FORTRAN

a[1..k, 1..m, 1..n]

a_{111} a_{211} a_{311} … a_{k11} a_{*11}
a_{121} a_{221} a_{321} … a_{k21} a_{*21}
……………………………
a_{1m1} a_{2m1} a_{3m1} … a_{km1} a_{*m1}
a_{**1}

a_{112} a_{212} a_{312} … a_{k12}
a_{122} a_{222} a_{322} … a_{k22}
……………………………
a_{1m2} a_{2m2} a_{3m2} … a_{km2}
a_{**2}

a_{11n} a_{21n} a_{31n} … a_{k1n}
a_{12n} a_{22n} a_{32n} … a_{k2n}
……………………………
a_{1mn} a_{2mn} a_{3mn} … a_{kmn}

- C++ multidimensional array

ELEM A[d_1][d_2]…[d_n];

$$loc(A[j_1, j_2, \ldots, j_n]) = loc(A[0, 0, \ldots, 0])$$
$$+d \cdot [\, j_1 \cdot d_2 \cdot \ldots \cdot d_n + j_2 \cdot d_3 \cdot \ldots \cdot d_n$$
$$+ \ldots + j_{n-1} \cdot d_n + j_n \,]$$
$$= loc(A[0, 0, \ldots, 0]) + d \cdot [\sum_{i=1}^{n-1} j_i \prod_{k=i+1}^{n} d_k + j_n]$$

Special Matrices Implemented by Arrays

- Triangular matrix (upper/lower)
- Symmetric matrix
- Diagonal matrix
- Sparse matrix

Lower Triangular Matrix

- One-dimensional array: list[0.. $(n^2+n)/2-1$]
  - The matrix element $a_{i,j}$ is stored in list[$(i^2+i)/2 + j$] $(i>=j)$

$$\begin{pmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ 7 & 5 & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ 9 & 0 & 0 & 1 & 8 & \\ 0 & 6 & 2 & 2 & 0 & 7 \end{pmatrix}$$

Symmetric Matrix

$$\begin{bmatrix} 0 & 3 & 0 & 15 \\ 3 & 0 & 4 & 0 \\ 0 & 4 & 0 & 6 \\ 15 & 0 & 6 & 0 \end{bmatrix}$$

- Satisfies that $a_{i,j} = a_{j,i}$, $0 \le i, j < n$

  The matrix on the right is a (symmetric) adjacent matrix for a undirected graph

- Store the lower triangle in a 1-dimensional array

  sa[0..n (n+1) /2-1]

  - There is a one-to-one mapping between sa[k] and $a_{i,j}$:

$$k = \begin{cases} j(j+1)/2 + i, i < j \\ i(i+1)/2 + j, i \ge j \end{cases}$$

Diagonal Matrix

- Diagonal matrix: all non-zero elements are located at diagonal lines.
- Band matric: a[i][j] = 0 when |i-j| > 1
  - A band matrix with bandwidth 1 is shown as below

$$
\begin{pmatrix}
a_{0,0} & a_{0,1} & & \cdots\cdots & 0 \\
a_{1,0} & a_{1,1} & a_{1,2} & \cdots\cdots & \\
 & & & & \\
\cdots\cdots & & & & a_{n-2,n-1} \\
0 & & & a_{n-1,n-2} & a_{n-1,n-1}
\end{pmatrix}
$$

Sparse Matrix

- Few non-zero elemens, and these elements distribute unevenly

$$\mathbf{A}_{6\times 7} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

12.1 Multidimensional Array

- Sparse Factor
  - In a m×n matrix, there are t non-zero elements, and the sparse factor is:

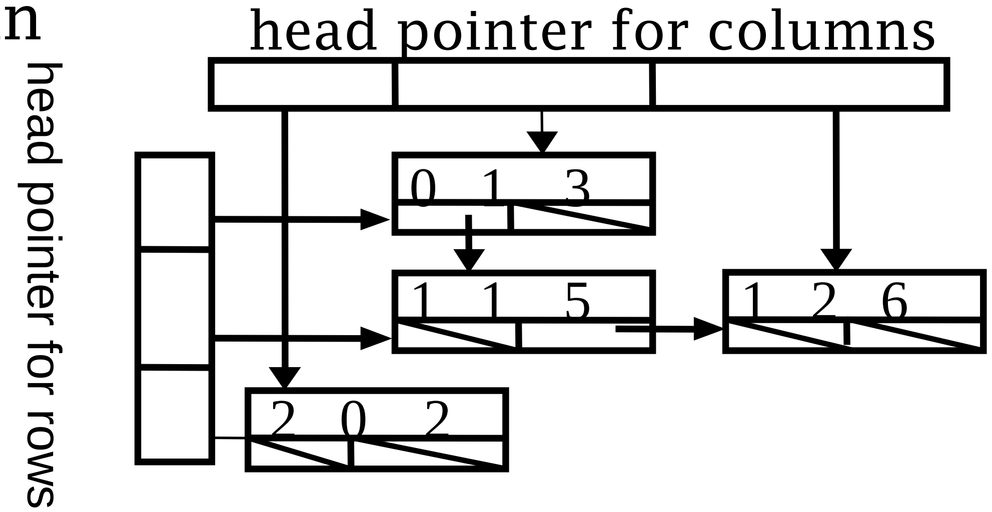
$$\delta = \frac{t}{m \times n}$$

  - When this value is lower than 0.05, the matrix could be considered a sparse matrix.
- 3-tuple (i, j, a_{ij}): commonly used for input/output
  - i is the row number
  - j is the column number
  - a_{ij} is the element value

Orthogonal Lists of a Sparse Matrix

- An orthogonal list consists of two sets of lists
  - pointer sequense for rows and columns
  - Each node has two pointers: one points to the successor on the same row; the other points to the successor on the same column

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 5 & 6 \\ 2 & 0 & 0 \end{bmatrix}$$



Classic Matrix Multiplication

- A[c1..d1][c3..d3] , B[c3..d3][c2..d2], C[c1..d1][c2..d2]。

$$C = A \times B \quad (C_{ij} = \sum_{k=c3}^{d3} A_{ik} \cdot B_{kj})$$

-

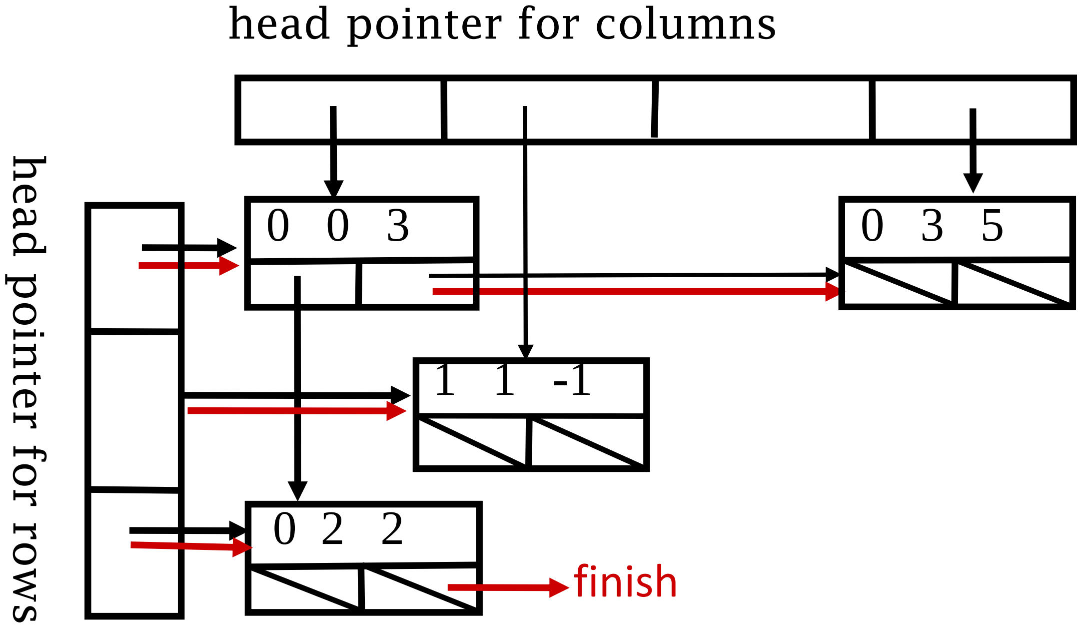
Time Cost of Classic Matrix Multiplication

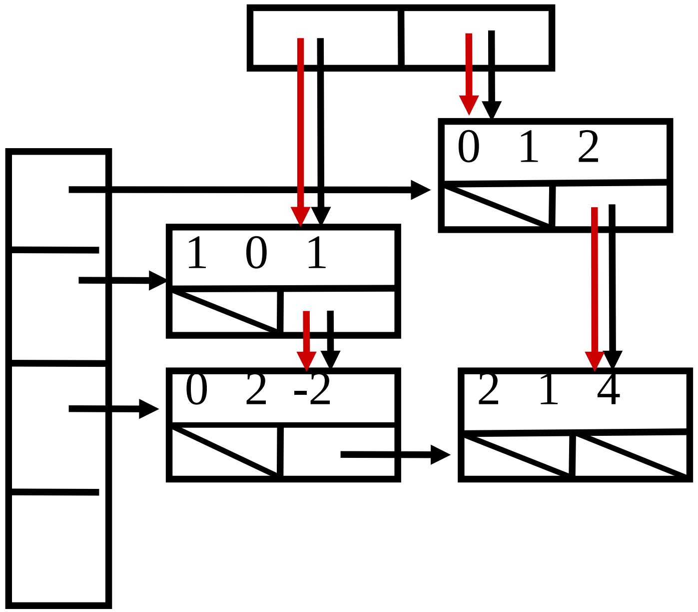
- $p=d1-c1+1$, $m=d3-c3+1$, $n=d2-c2+1$;
- A is a $p \times m$ matrix, B is a $m \times n$ matrix, resulting in C, a $p \times n$ matrix
- So the time cost of the classic matrix multiplication is $O(p \times m \times n)$

```
for  (i=c1; i<=d1; i++)
    for  (j=c2; j<=d2; j++){
        sum = 0;
        for  (k=c3; k<=d3; k++)
            sum = sum + A[i,k]*B[k,j];
        C[i , j] = sum;
    }
```

Sparse Matrix Multiplication

$$\begin{bmatrix} 3 & 0 & 0 & 5 \\ 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ -2 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & \mathbf{6} \\ \mathbf{-1} & 0 & \\ 0 & 4 & \mathbf{4} \end{bmatrix}$$





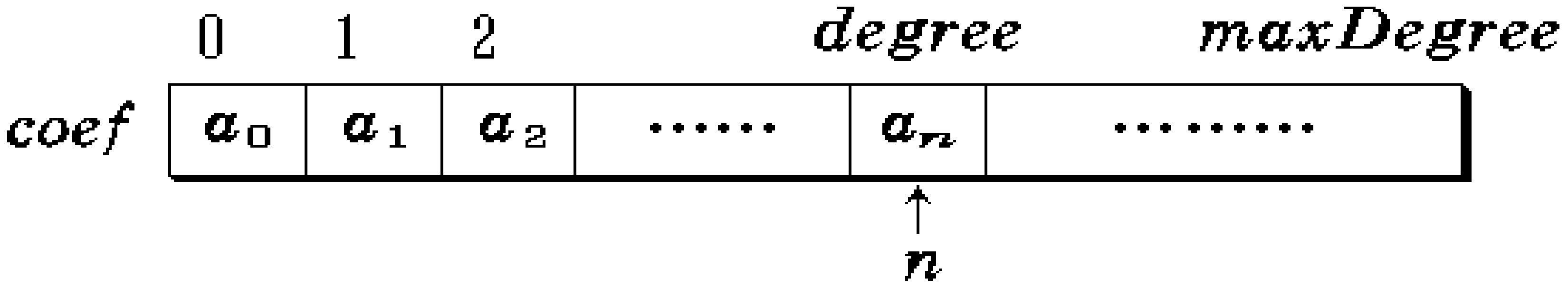
Time Cost of Sparse Matrix Multiplication

- A is a p×m matrix, B is a m×n matrix, resulting in C, a p×n matrix.
  - If the number of non-zero elements in a row of A is at most t_a
  - and the number of non-zero elements in a column of B is at most t_b
- Overall running time is reduced to O ((t_a+t_b) ×p×n)
- Time cost of classic matrix multiplication is O (p×m×n)

Applications of Sparse Matrix

polynomial of one indeterminate

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$
$$= \sum_{i=0}^{n} a_i x^i$$



Chapter 12 Advanced Data Structure

- 12.1 Multi-array
- 12.2 Generalized List
  - Basic Concepts
  - Different Types of Generalized List
  - Storage of Generalized List
  - Traversal algorithm for Generalized List
- 12.3 Storage management
- 12.4 Trie
- 12.5 Improved BST

Basic Concepts

- Review of linear list
  - Finite ordered sequence consisting of n(>=0) elements.
  - All elements of a linear list have the same type.
- If a linear list contains one or more sub-lists, then it is called a generalized list, usually represented as:
  - $L = (x_0, x_1, \ldots, x_i, \ldots, x_{n-1})$

$$L= (x_0, x_1, \ldots, x_i, \ldots, x_{n-1})$$

- L is the **name** of this generalized list.
- n is the **length.**
- Each $x_i(0 \le i \le n-1)$ is an **element.**
  - either a single element, i.e. atom,
  - or another generalized list, i.e. sublist.
- **Depth** : the number of brackets when all the elements are converted to atoms.

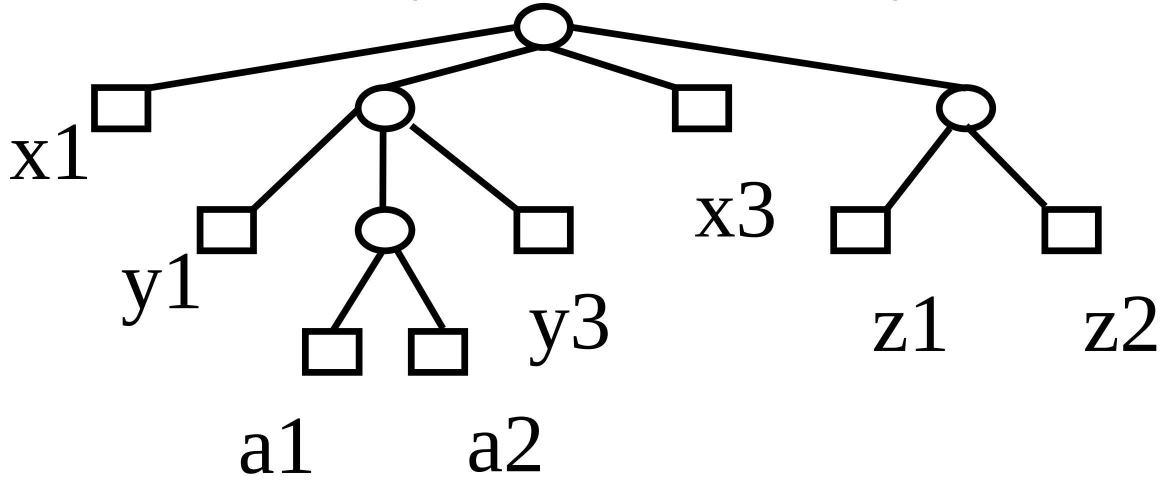
L= (x_0, x_1, …, x_i, …, x_{n-1})

- head = x_0
- tail = (x_1, …, x_{n-1})
  - smaller lists
- Easier to store and to implement.

Different Types of Generalized Lits

- pure list
  - There is only one path existing from root to each leaf.
  - i.e. each element (atom, sublist) only appears once.

(x1, (y1 , (a1 ,a2) , y3) , x3 , (z1 ,z2))

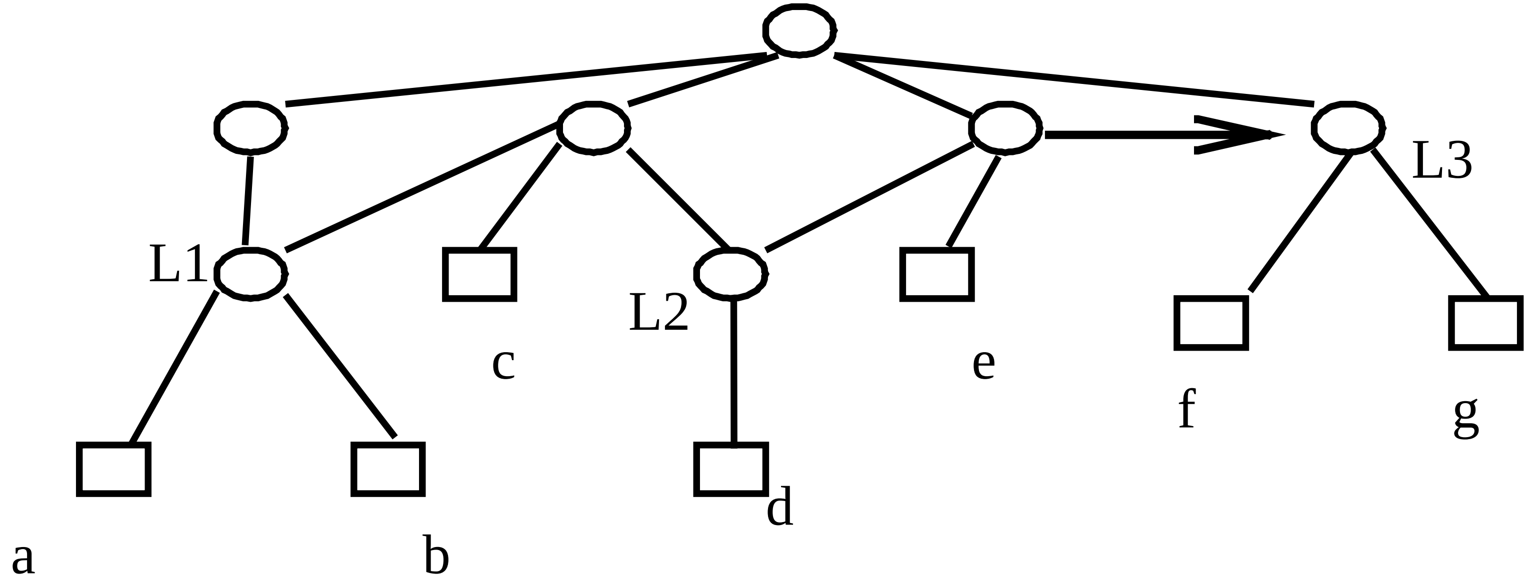


Different Types of Generalized Lits

- Reentrant lists
  - Its elements (atoms or sublists) might appear more than once.
  - Corresponds to a DAG if no circles exists.
- Sublists and atoms are labeled.

e.g. cycle lists

(((a, b)) , ((a,b) ,c,d) , (d, e, f, g) , (f ,g))

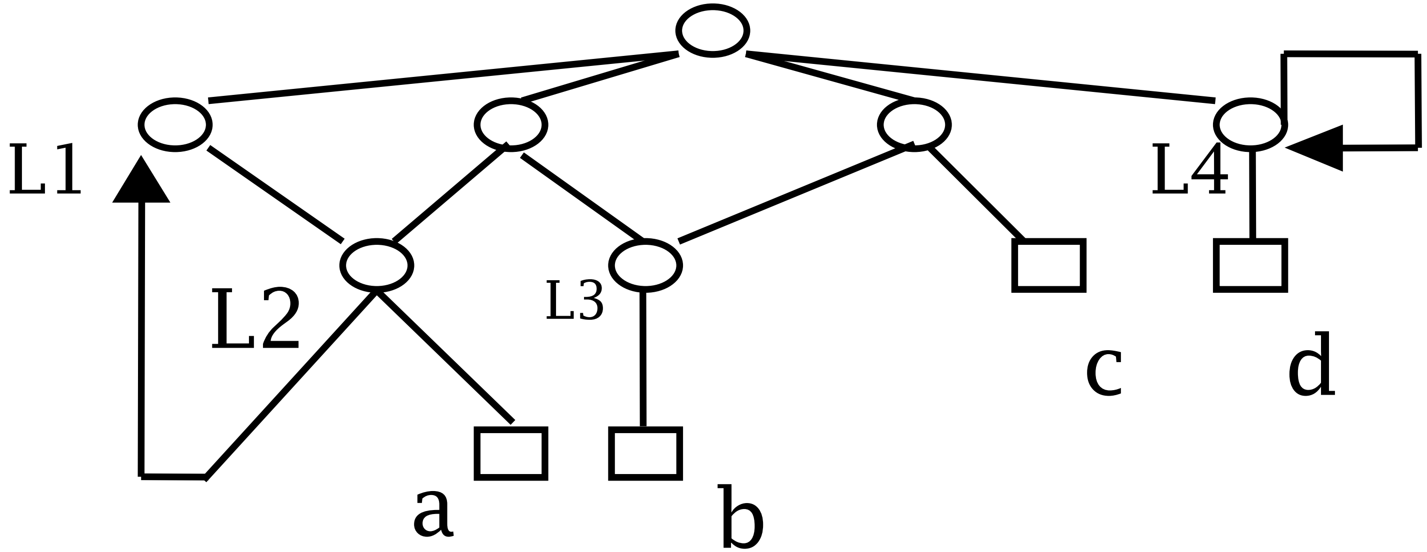


(L1: (a,b) , (L1, c ,L2: (d)) , (L2, e,L3: (f,g)) , L3)

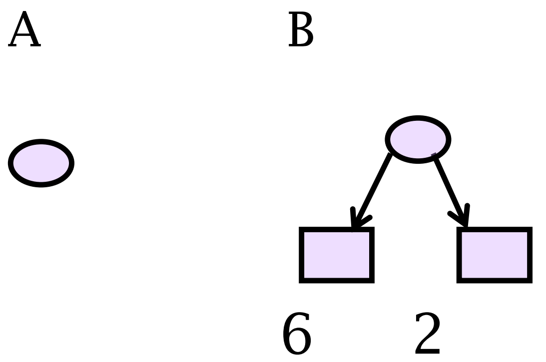
Different Types of Generalized Lits

- Circle lists
  - contains circles.
  - with infinite depth.

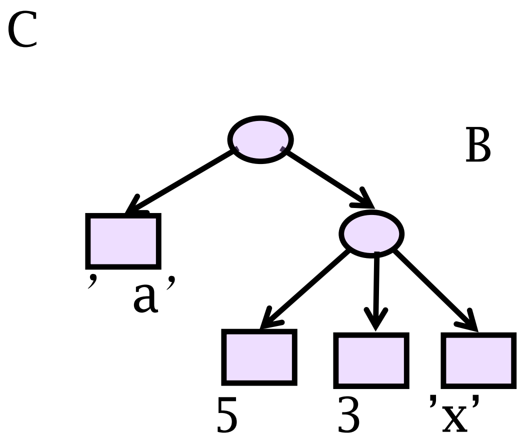
(L1: (L2: (L1, a)) , (L2, L3: (b)) , (L3, c) , L4: (d,L4))

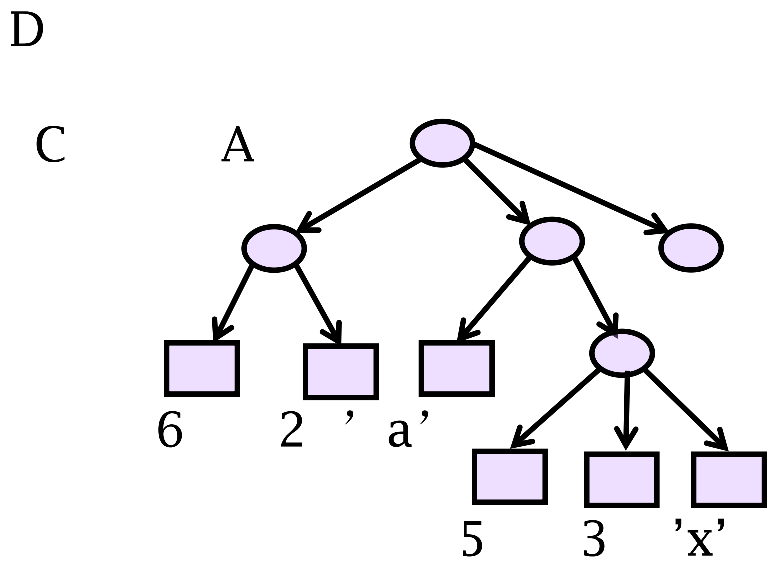


12.2 Generalized list and Storage management

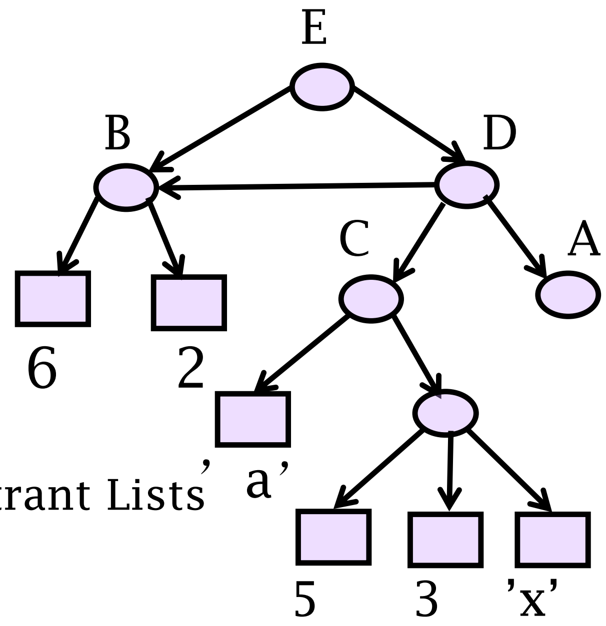


Linear Lists

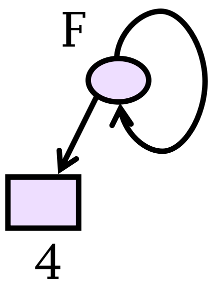




Pure Lists



Reentrant Lists



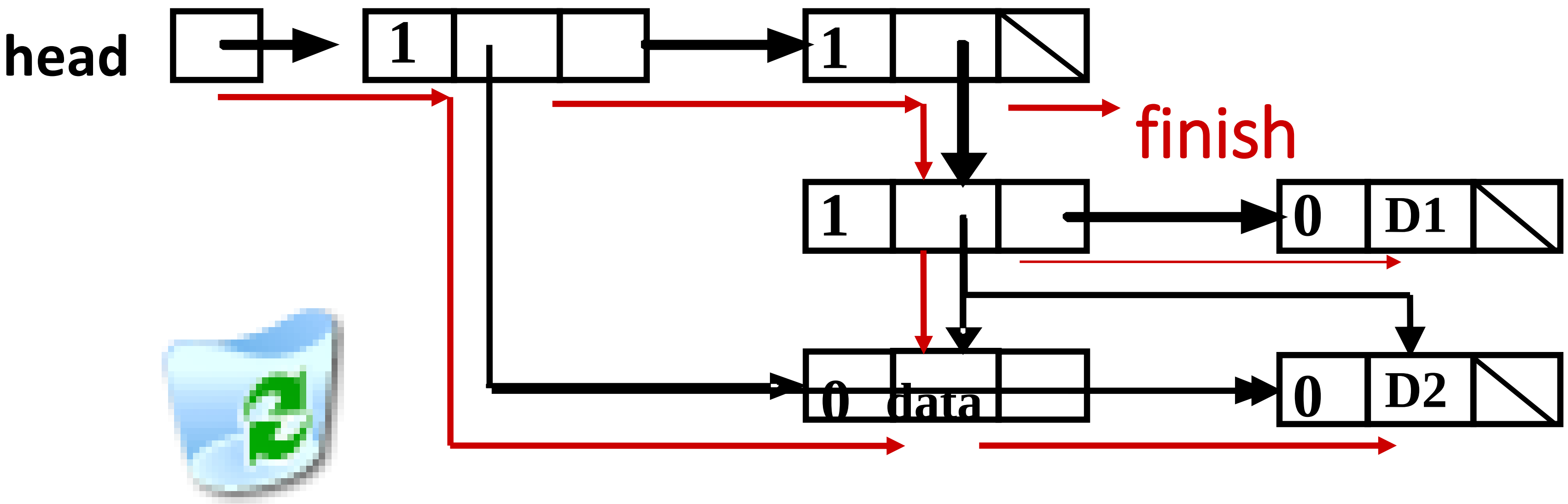
Circle Lists

12.2 Generalized list and Storage management

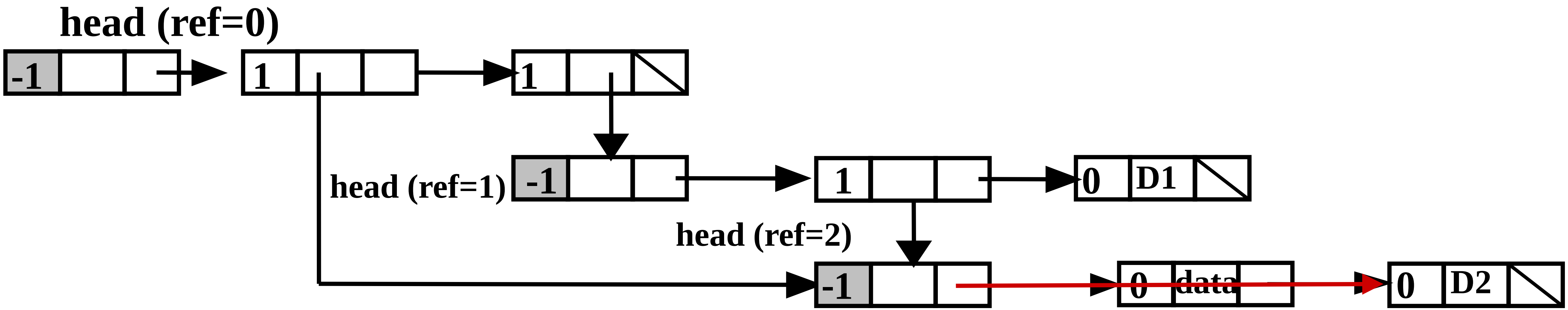
- Graph ⊇ Reentrant List ⊇ Pure List(Tree) ⊇ Linear List
    - Generaized lists are extensions of linear and tree structures.
- Circle lists are reentrant lists that have circles.
- Applications of generalized lists
    - Relations between the invocation of the function
    - Reference relations in memory space
    - LISP

Storage of Generalized Lists

- Generalized link lists without head node
  - Problems might occur when deleting nodes.
  - The list must be adjusted when deleting node 'data'.

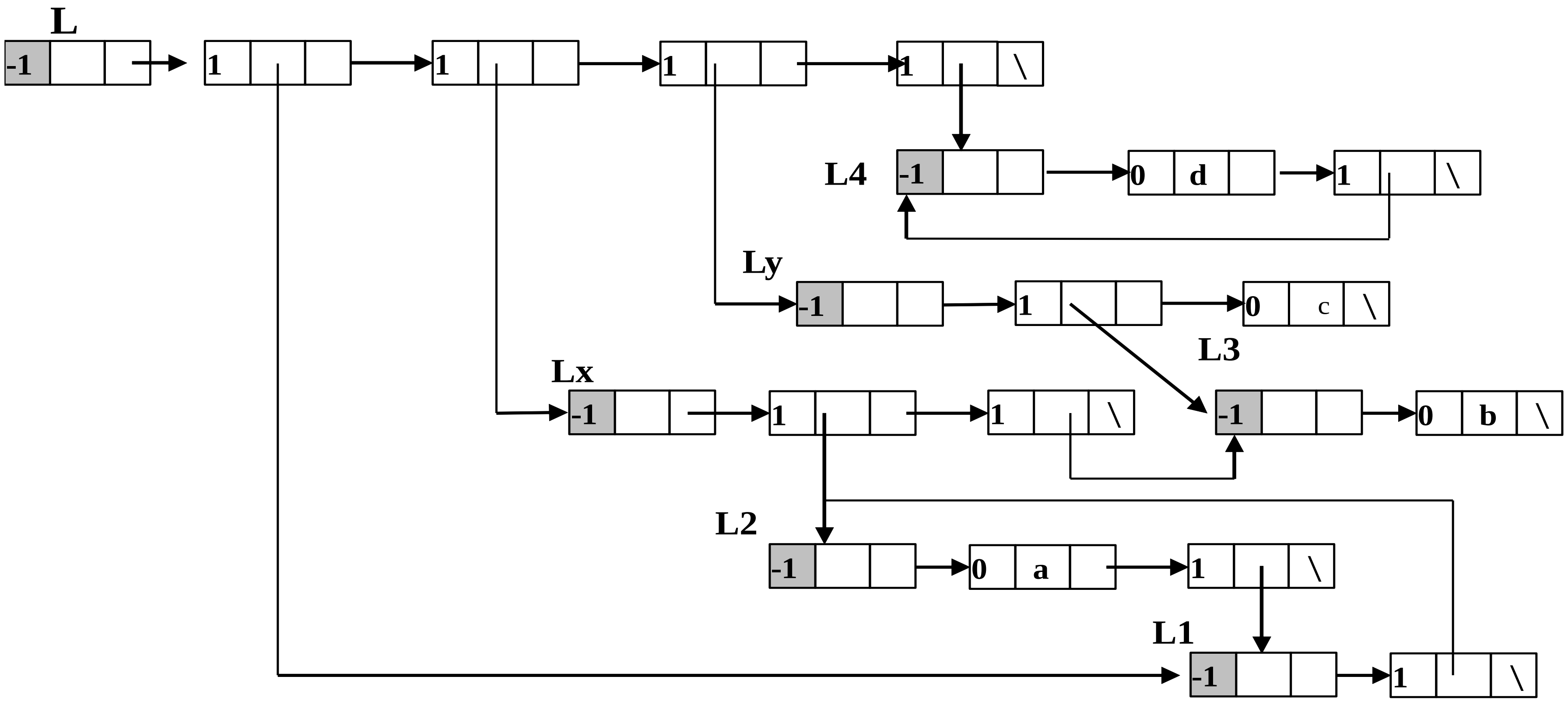


Storage of Generalized Lists



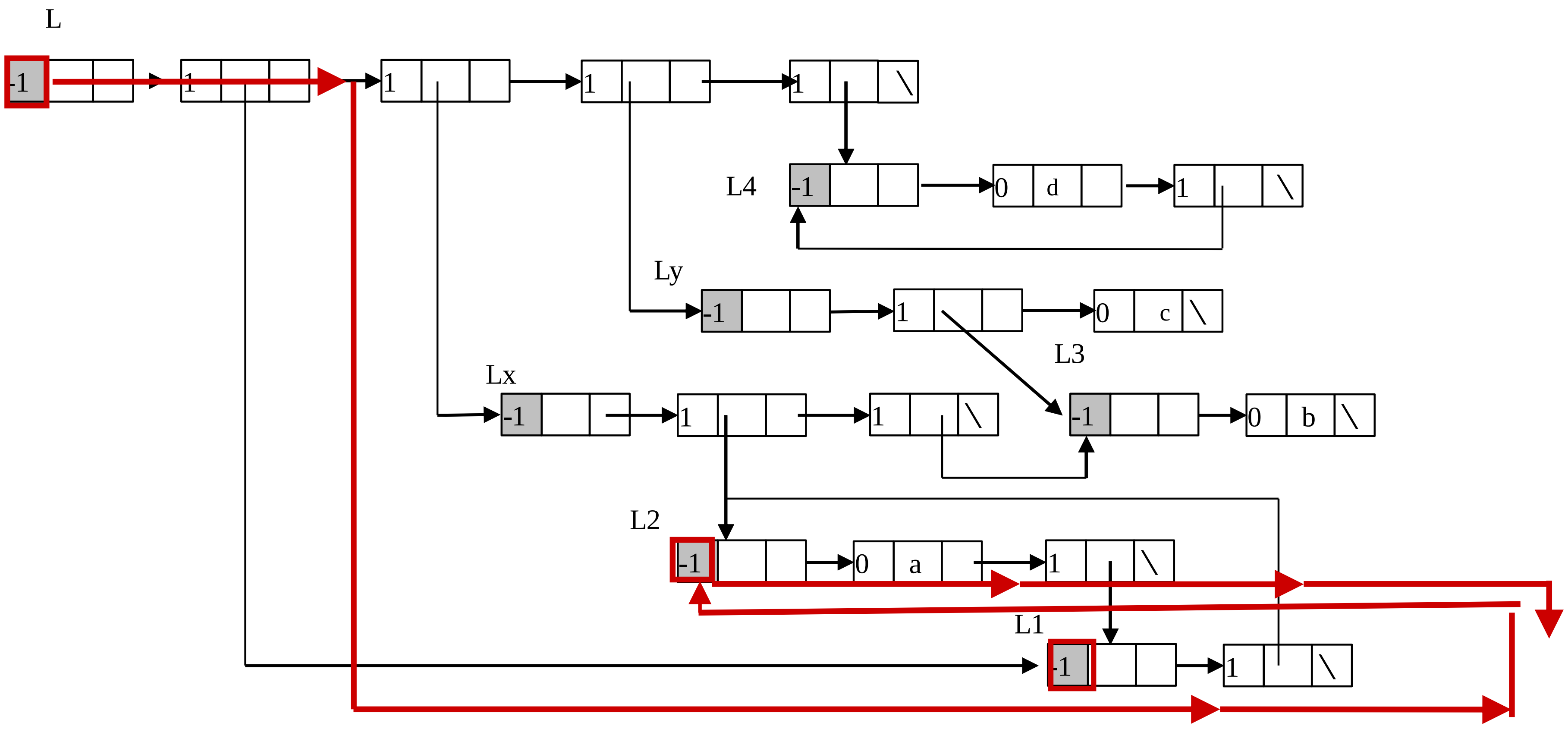
- Add the head node, and the deleting/inserting operation would be simplified.
- Reentrant lists, especially circle lists
  – mark each node (because it is a graph)

Circle Generalized Lists with Head Nodes



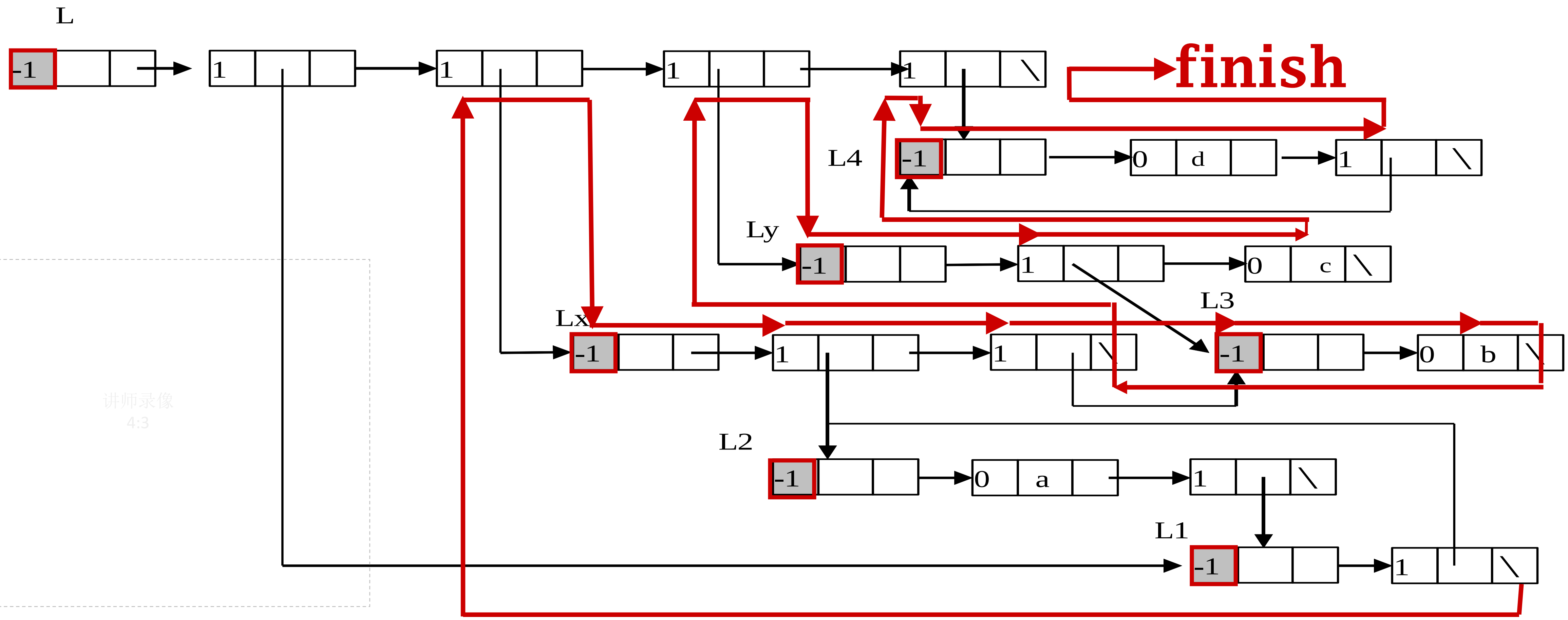
12.2 Generalized list and Storage management

(L1 : (L2 : (a , L1)))



12.2 Generalized list and Storage management

(L1: (L2: (a ,L1)) , Lx : (L2 , L3 : (b)) , Ly : (L3 , c) , L4 : (d , L4))



Chapter 12 Advanced Data Structure

- 12.1 Multidimensional array
- 12.2 Generalized Lists
- 12.3 Storage management
  - Allocation and Reclamation
  - Freelist
  - Dynamic Memory Allocation and Reclamation
  - Failure Policy and Collection of Useless Units
- 12.4 Trie
- 12.5 Improved BST

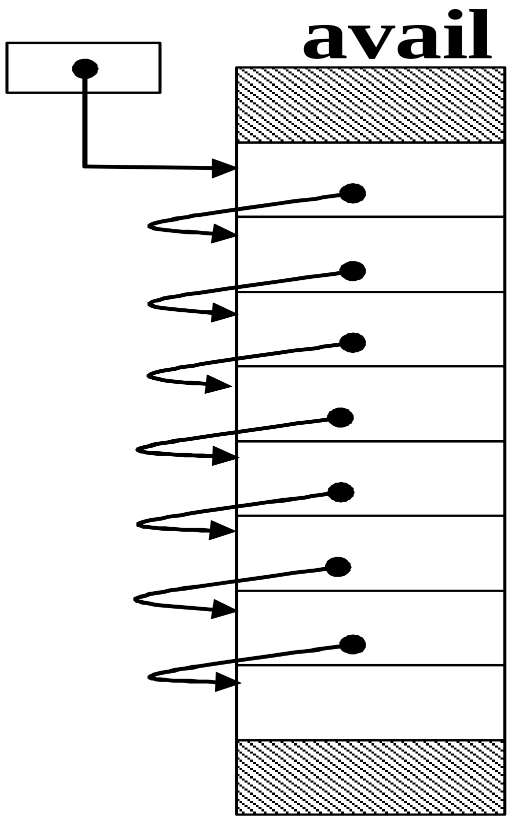
Allocation and Reclamation

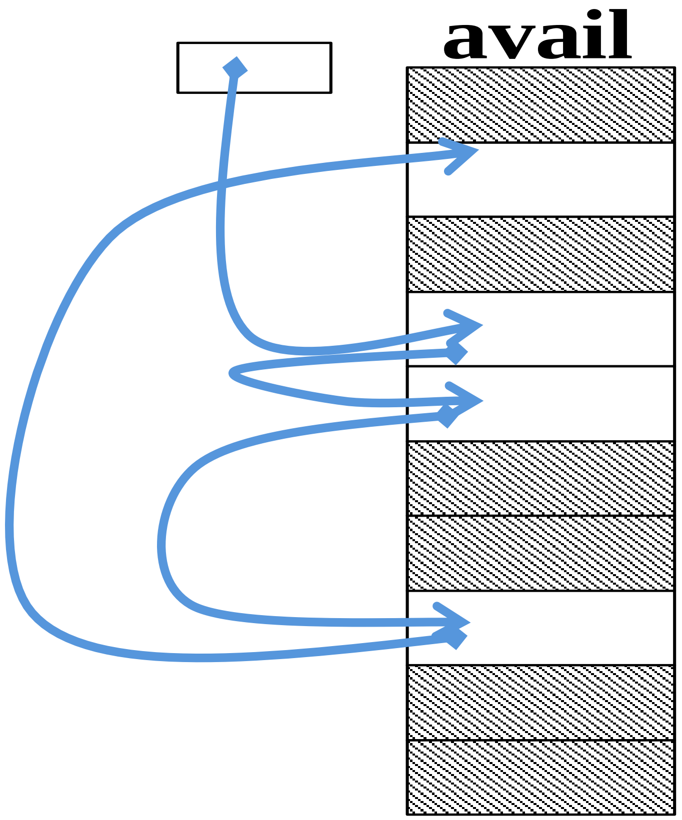
- Basic problems in storage management
  - Allocate memory
  - Reclaim "freed" memory
- Fragmentation problem
  - The compression of storage
- Collection of useless units
  - Useless units: memory that can be collected but has not been collected yet
  - Memory leak
    - Programmers forget to delete pointers which will not be used

Freelist

- Consider the memory as an array of changeable number of blocks
  - Some blocks has been allocated
  - Link free blocks together, and form a freelist.
- Memory allocation and reclamation
  - new p: allocate from available space
  - delete p: return the block that p points to to the freelist.
- If there is not enough space, resort to failure policy.

12.3 Storage Management





(1) initial state of the freelist

(2) freelist after the system has run for a while

freelist with nodes of equal length

Function overloading of freelist

```
template <class Elem> class LinkNode{
private:
    static LinkNode  avail;             // head pointer
public:
    Elem value;                         // value of each node
    LinkNode   next;                    // pointer pointing to next node
    LinkNode (const Elem & val, LinkNode   p) ;
    LinkNode (LinkNode   p = NULL) ;    // construction function
    void   operator new (size_t) ;      // redefine new
    void operator delete (void   p) ;   // redefine delete
};
```

12.3 Storage Management

```
//implementation of new
template <class Elem>
void   LinkNode<Elem>::operator new (size_t) {
    if (avail == NULL)              //if the list is empty
        return ::new LinkNode;      //allocate memory using new
    LinkNode<Elem>   temp = avail;
                                    //allocate from available space
list
    avail = avail->next;
    return temp;
}
```

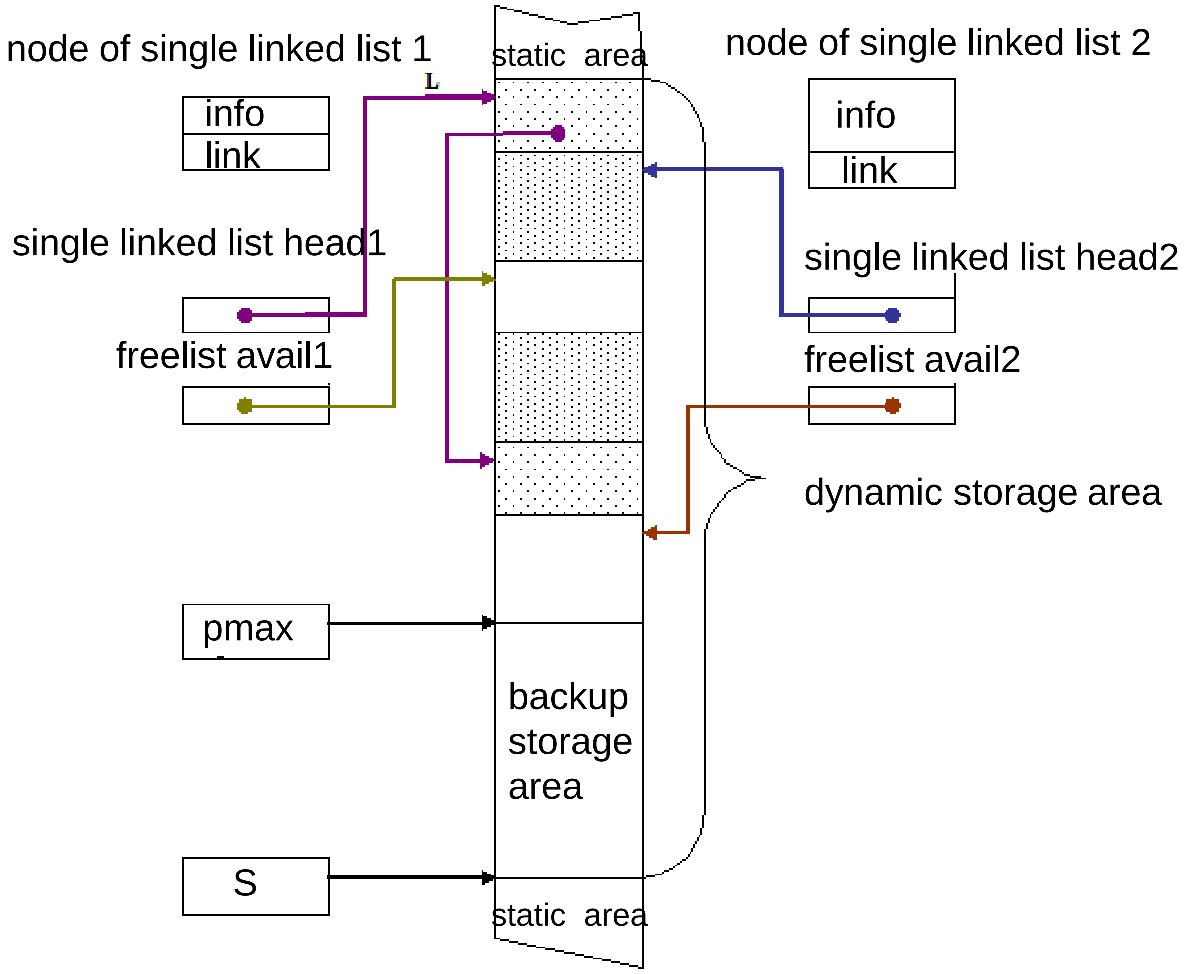
12.3 Storage Management

```
//implementation of delete
template <class Elem>
void LinkNode<Elem>::operator delete (void   p) {
    ( (LinkNode<Elem>   )  p) ->next = avail;
    avail =  (LinkNode<Elem>   ) p;
}
```

Free List: Stack in a Singly-Linked List

- new: deletion in the stack
- delete: insertion in the stack
- If the default new and delete operations are needed, use **"::new p" and "::delete p".**
  - For example, when a program is finished, return the memory occupied by avail back to the system (free the memory completely)

12.3 Storage Management



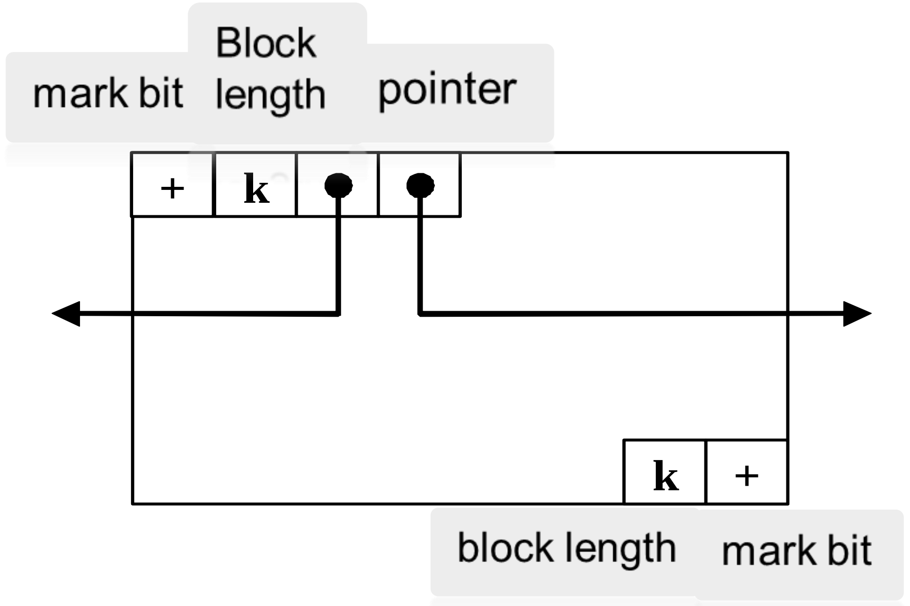
- When pmax is equal to or larger than S, no more memory can be allocated.

Dynamic Memory Allocation and Reclamation

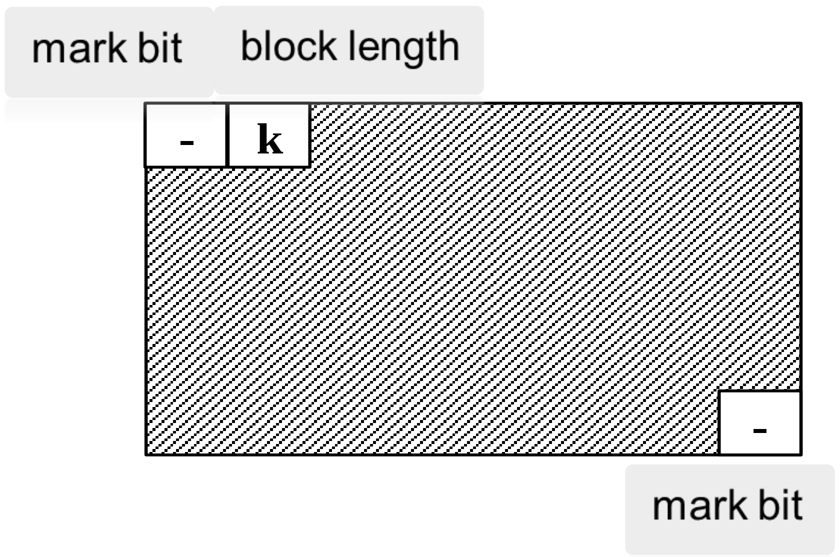
Available blocks with variable lengths

- Allocation
  - Find a block whose length is larger than the requested length.
  - Truncate suitable length from it.
- Reclamation
  - Consider whether the space deleted can be merged with adjacent nodes,
  - So as to satisfy later request of large node.

Data Structure of Free Blocks

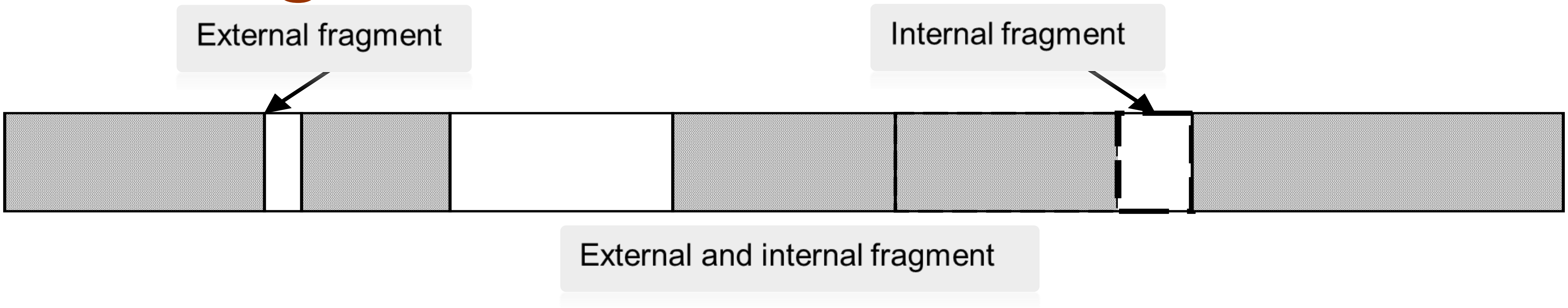


(a) structure of free block



(b) structure of allocated block

Fragmentation Problem



- Internal fragment: space larger than the requested bytes
- External fragment: small free blocks

Sequential Fit

Allocation of free blocks

- Common sequential fit algorithms
  - first fit
  - best fit
  - worst fit

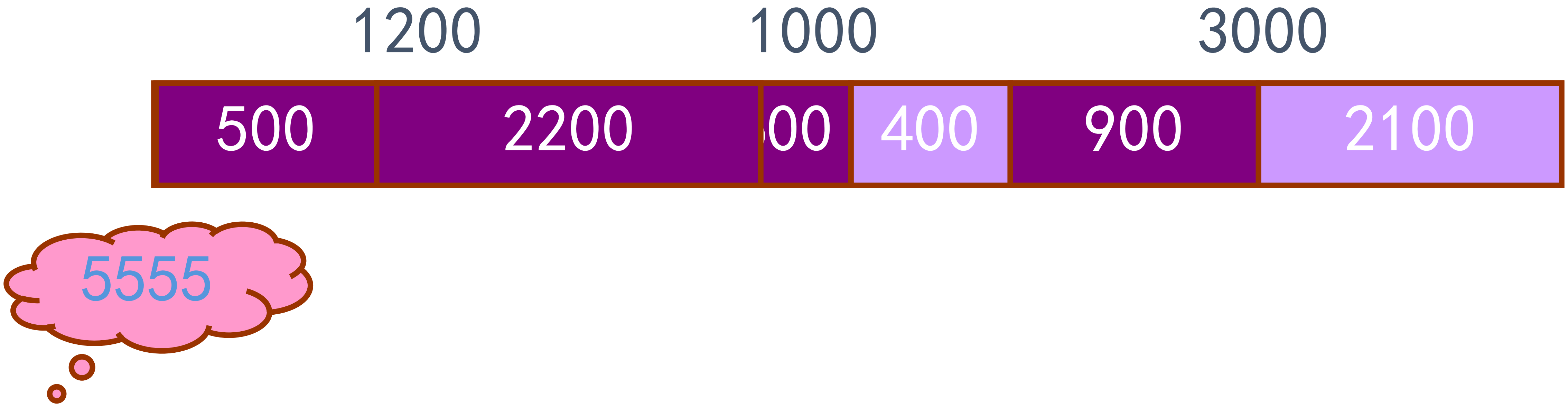
Sequential Fit

1200			1000		3000	
600	500	100	900	100	2200	800

- 3 Blocks 1200， 1000， 3000
  request sequence: 600, 500, 900, 2200
- first fit:

Sequential Fit

- best fit



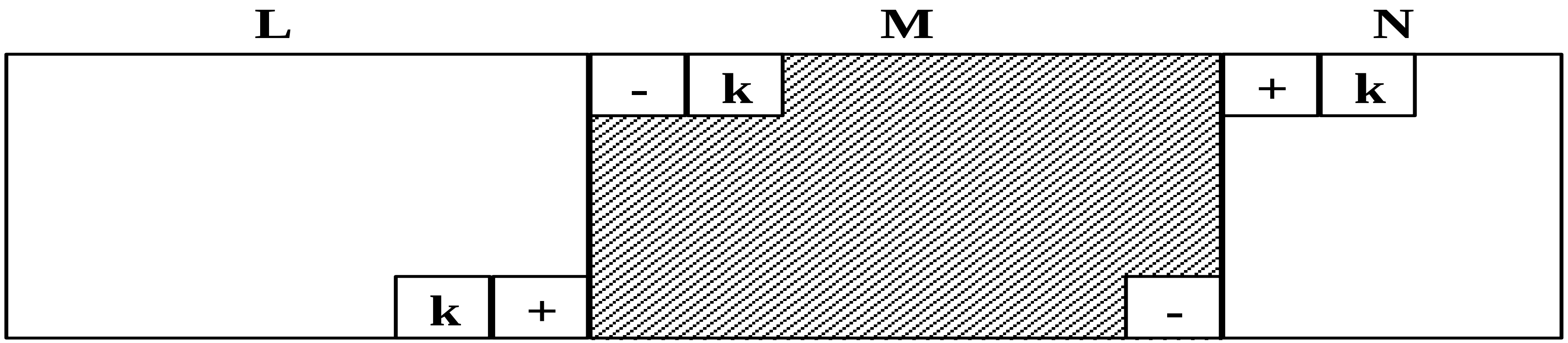
request sequence: 600, 500, 900, 2200

Sequential Fit

- worst fit

request sequence: 600, 500, 900, 2200

Reclamation: merge adjacent blocks



allocate block M back to the freelist

Fitting Strategy Selection

- Need to take the following user request into account
  - Importance of allocation and reclamation efficiency.
  - Variation range of the length of allocated memory
  - Frequency of allocation and reclamation
- In practice, fist fit is **the most commonly used.**
  - Quicker allocation and reclamation.
  - Support random memory requests.

Hard to decide which one is the best in general.

Failure Policy and Collection of Useless Units

- If a memory request cannot be satisfied because of insufficient memory, the memory manager has two options:
  - do nothing, and return failure info;
  - follow failure policy to satisfy requests.

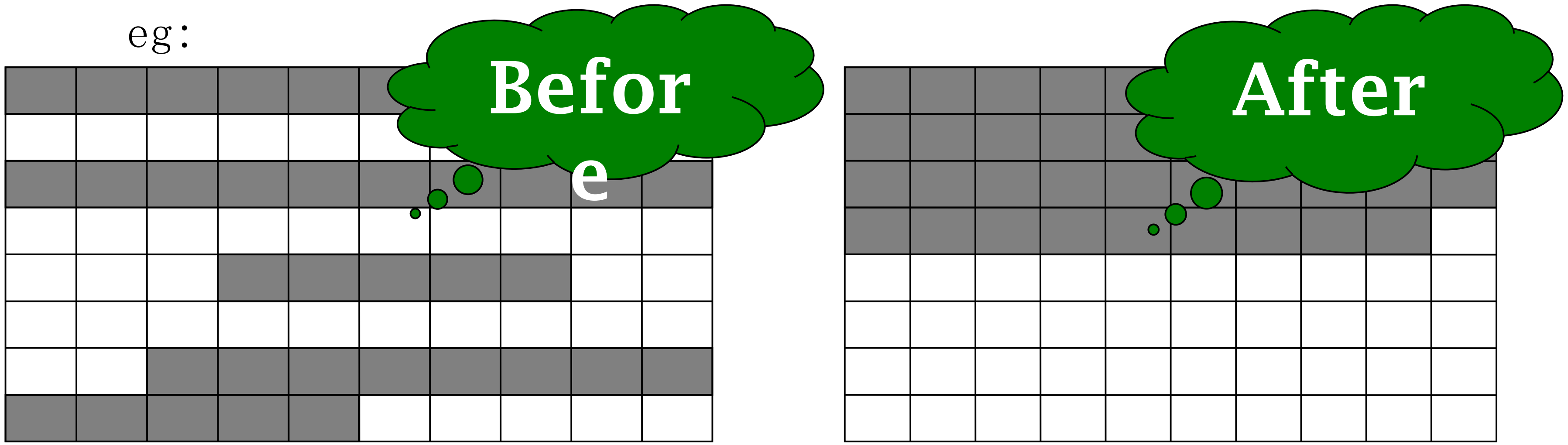
Compaction

- Collect all the fragments together
  - Generate a larger free block.
  - Used when there are a lot of fragments.
- Handler makes the address relative
  - Secondary indirect reference to the storage location.
  - Only have to change handlers to move blocks.
    - No need to change applications.

Two Types of Compaction

- Perform a compact once a block is freed.
- Perform a compact when there is not enough memory or when collecting useless units.

eg:



Collecting Useless Units

- Collecting useless units: the most complete failure policy.
  - Search the whole memory, and label those nodes not belonging to any link.
  - Collect them to the freelist.
  - The collection and compaction processes usually can perform at the same time.

Data Structures and Algorithms

Thanks

the National Elaborate Course (Only available for IPs in China)
http://www.jpk.pku.edu.cn/pkujpk/course/sjjg/

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