SOLUTIONS to final exam
Principles of Economics with Calculus
Caltech/edX
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Question 1: Solution

- Consumer's problem before change:

  $$\max_{x \geq 0} 8\sqrt{x + sx} - x$$

  The FOC for this problem is

  $$\frac{4(1+s)}{\sqrt{x+sx}} - 1 = 0$$

  Solving for x yields

  $$x^* = 16(1+s)$$

  Consumer's net utility change from buying optimal quantity of x:

  $$8\sqrt{16(1+s)^2} - 16(1+s) = 16(1+s)$$

- Consumer's problem after switch to rebate:

  $$\max_{x \geq 0} 8\sqrt{x} - x + rx$$

  FOC:

  $$\frac{4}{\sqrt{x}} - (1-r) = 0$$

  Solving for x yields

  $$x^* = \frac{16}{(1-r)^2}$$

  Consumer's net utility from buying optimal quantity after change to rebate:

  $$8\sqrt{\frac{16}{(1-r)^2}} - \frac{16}{(1-r)^2}(1-r) = \frac{16}{1-r}$$

- For consumer to be indifferent between these two policies, they must allow the consumer to achieve the same net utility form purchasing an optimal quantity of x. Therefore the desired r must solve

  $$16(1+s) = \frac{16}{1-r}$$

  which yields

  $$r = \frac{s}{1+s}$$

Question 2: Solution

- Denote tax rate by τ. Then consumer demand with the tax in place is given by

$$1000 - 10(p + \tau)$$

and the market equilibrium is:

$$10p = 1000 - 10(p + \tau) \implies p^*(\tau) = 50 - \frac{\tau}{2}$$

$$q^*(\tau) = 10p^* = 500 - 5\tau$$

Government revenue at tax rate τ is given by

$$\tau q^*(\tau) = 500\tau - 5\tau^2$$

FOC:

$$500 - 10\tau = 0 \implies \tau^* = 50$$

So maximum government revenue is

$$50q^*(50) = 50(500 - 5(50)) = 12500$$

Question 3: Solution

- Quantity sold in market A dictated by government, so focus on market B.
- Monopolist's problem:

$$\max_{q_B \geq 0}(1000 - q_B)q_B - (250 + q_B)^2$$

FOC:

$$1000 - 2q_B - 2(250 + q_B) = 0 \implies q_B^* = 125$$

- Total produced $= 250 + 125 = 375$.

Question 4: Solution

- Firm i's problem:

$$\max_{q_i \geq 0}(1000 - q_i - q_j)q_i - q_i^2 + 100q_i$$

FOC:

$$1000 - 2q_i - q_j - 2q_i + 100 = 0 \implies q_i^*(q_j) = \frac{1100 - q_j}{4}$$

Firm's are identical, so there is a symmetric equilibrium where $q_i^* = q_j^*$,

$$q_i^*(q_i) = \frac{1100 - q_i}{4} \implies \frac{5}{4}q_i = \frac{1100}{4} \implies q_i^* = 220$$

Thence $q_j^* = q_i^* = 220$, and the total quantity produced in the market is 440.

- To find equilibrium price, plug equilibrium quantity into demand function:

$$440 = 1000 - p \implies p^* = 560$$

Question 5: Solution

- Begin as before, but with s instead of 100 for subsidy/tax. Firm i's problem:

  $$\max_{q_i \geq 0} (1000 - q_i - q_j)q_i - q_i^2 + sq_i$$

  FOC:

  $$1000 - 2q_i - q_j - 2q_i + s = 0 \implies q_i^*(q_j) = \frac{1000 + s - q_j}{4}$$

  Firm's are identical, so

  $$q_i^* = q_i^*(q_i) = \frac{1000 + s - q_i}{4} \implies \frac{5}{4}q_i = \frac{1000 + s}{4} \implies q_i^* = 200 + \frac{s}{5}$$

  Therefore market equilibrium is $Q^* = 400 + \frac{2s}{5}, p^* = 600 - \frac{2s}{5}$

- Now can compute CS as a function of s:

  $$CS(s) = \underbrace{\int_0^{400+\frac{2s}{5}} 1000 - q - \left(600 - \frac{2s}{5}\right) dq}_{\text{Net benefit from buying}} - \underbrace{\left(400 + \frac{2s}{5}\right) s}_{\substack{\text{cost of subsidy/} \\ \text{revenue from tax}}}$$

  $$= \frac{1}{2}\left(400 + \frac{2s}{5}\right)^2 - \left(400 + \frac{2s}{5}\right) s$$

- Optimal s is solution to

  $$\max_s CS(s)$$

  FOC:

  $$\frac{2}{5}\left(400 + \frac{2s}{5}\right) - \left(400 + \frac{2s}{5}\right) - \frac{2s}{5} = 0$$

  $$\implies s^* = -375$$

Question 6: Solution

- Let q be the quantity each consumer consumes. Optimal q solves

  $$\max_{q \geq 0} 100 \left(\frac{1}{2}\sqrt{q} - q + \sqrt{100q}\right)$$

FOC (after cancelling factor of 100):

$$\frac{1}{4\sqrt{q}} - 1 + \frac{100}{2\sqrt{100q}} = 0$$

$$\implies q^{opt} = \left(\frac{21}{4}\right)^2$$

Total optimal quantity $Q^{opt} = 100q^{opt} = \left(\frac{210}{4}\right)^2$

- In market equilibrium, each consumer's q solves

$$\max_{q \geq 0} \frac{1}{2}\sqrt{q} - q$$

FOC:

$$\frac{1}{4\sqrt{q}} - 1 = 0$$

$$\implies q^* = \frac{1}{16}$$

$$\implies Q^* = \frac{100}{16}$$

- Difference of total optimal quantity minus total market equilibrium quantity given by

$$Q^{opt} - Q^* = \frac{44100}{16} - \frac{100}{16} = 2750$$

1 Question 7: Solution

- Demand function is solution to consumer's problem,

$$\max_{q \geq 0} \frac{1}{2}\sqrt{q} - pq$$

FOC:

$$\frac{1}{4\sqrt{q}} - p = 0$$

$$\implies q^D = \frac{1}{16p^2}$$

$$\implies Q^D = \frac{100}{16p^2}$$

$$\implies p = \frac{10}{4\sqrt{Q}}$$

- Total quantity sold is then solution to monopolist's problem,

$$\max_{q\geq 0} \frac{10}{4\sqrt{q}}q - q$$

FOC:

$$\frac{5}{4\sqrt{q}} - 1 = 0$$

$$\implies q^{mon} = \frac{25}{16}$$

Note that since this is the optimal quantity supplied by a monopolist, it is the same as the total quantity bought in the market.

- Difference is

$$Q^{opt} - q^{mon} = \frac{44100}{16} - \frac{25}{16} = 2754.6875$$

Question 8: Solution

- Fixed costs must be paid regardless, so we can ignore them. Then firm's cost function is

$$c(q) = \begin{cases} 0 & q = 0 \\ 1000 + q^2 & q > 0 \end{cases}$$

Firm's problem is

$$\max_{q\geq 0} pq - c(q)$$

If the solution is interior, it must solve the FOC

$$p = c'(q)$$

$$\implies q^* = \frac{p}{2}$$

Profits from producing q^* are given by

$$pq^* - c(q^*) = \frac{p^2}{2} - \left(\frac{p^2}{4} + 1000\right) = \frac{p^2}{4} - 1000$$

Profits from producing zero are zero.

- Firm will supply zero if

$$\frac{p^2}{4} - 1000 \leq 0$$

$$\implies p \leq \sqrt{4000}$$