Example of CLT for Symmetric Laminate with Mechanical Loading

These are two problems which detail the process through which the laminate deformations are predicted using Classical Lamination theory. The first only includes mechanical loads on a symmetric laminate; the second analyzes an asymmetric laminate and incorporates the effects of moisture and temperature changes.

Introductory remarks

The below is an example of all the calculations necessary for a general case to calculate the deformation of a laminate undergoing mechanical loading, temperature changes and moisture absorption. Recognizing special cases can reduce the calculations necessary to perform the predictions, e.g. if there is no moisture absorption, the entirety of those calculations can be neglected.

As well, if the laminate is symmetric, the B matrix does not need to be calculated, it will be 0 and if the laminate is balanced, $A_{16} = A_{26} = D_{16} = D_{26} = 0$. Similarly, if the change in temperature or moisture is zero, the fictitious forces and moments can be omitted. Finally, all the matrices are symmetric, thus only the upper triangle is calculated, the lower triangle is identical and therefore is simply reflected across the diagonal.

Problem Statement

If a (0/45/90/-45)$_S$ which is subjected to loading of

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} 300\,\frac{lb}{in} \\ 100\,\frac{lb}{in} \\ 0\,\frac{lb}{in} \\ 5\,\frac{lbin}{in} \\ 0\,\frac{lbin}{in} \\ 0\,\frac{lbin}{in} \end{bmatrix}$$

What are the resultant midplane strains, curvatures and effective properties?

Assume the lamina material properties are;

$$E_1 = 20 * 10^6\ psi\ E_2 = 1.5 * 10^6 psi, G_{12} = 1 * 10^6 psi, \nu_{12} = .34$$

$$\alpha_1 = 0.2 * 10^{-6} \frac{}{°F}, \alpha_2 = 20 * 10^{-6} \frac{}{°F}, \beta_1 = 0.2 * 10^{-4}, \beta_2 = 20 * 10^{-4}, thickness = 0.0075\ in$$

Solution:

Finding the Q and $\bar{Q}$ matrices

To solve this problem, we first need to find the Q matrix which is defined as:

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{62} & Q_{66} \end{bmatrix} = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

In order to evaluate this expression, we first need to find ν_{21}. From before, the expression is

$$\nu_{21} = \nu_{12} * \frac{E_2}{E_1} = 0.34 * \left(\frac{1.5}{20}\right) = 0.0255$$

Inserting the appropriate values we have:

$$[Q] = \begin{bmatrix} \frac{20*10^6}{1-0.34*0.0255} & \frac{0.34*1.5*10^6}{1-0.34*0.0255} & 0 \\ \frac{0.34*1.5*10^6}{1-0.34*0.0255} & \frac{1.5*10^6}{1-0.34*0.0255} & 0 \\ 0 & 0 & 1*10^6 \end{bmatrix} = \begin{bmatrix} 20.17*10^6 & 0.514*10^6 & 0 \\ 0.514*10^6 & 1.513*10^6 & 0 \\ 0 & 0 & 1*10^6 \end{bmatrix} psi$$

Then, for each unique ply, the $\overline{\mathrm{Q}}$ matrix needs to be found. The expressions are:

$$\bar{Q}_{11} = Q_{11}\cos^4\theta \; + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$

$$\bar{Q}_{12} = \bar{Q}_{21} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta)$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta \; + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta$$

$$\bar{Q}_{16} = \bar{Q}_{61} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta$$

$$\bar{Q}_{26} = \bar{Q}_{62} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^2\theta$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta)$$

From trigonometry, the 0° ply, $[\bar{Q}] = [Q]$, and for the 90° ply, since we have $\cos 90° = 0, \sin 90° = 1$, only the Q_{11} and Q_{22} expressions are reversed. Evaluating the expressions for $\theta = 0°, 90°, 45°, -45\,°$ the four $[\bar{Q}]$ matrices are thus:

$$[\bar{Q}]^0 = \begin{bmatrix} 20.17*10^6 & 0.514*10^6 & 0 \\ 0.514*10^6 & 1.513*10^6 & 0 \\ 0 & 0 & 1*10^6 \end{bmatrix}, [\bar{Q}]^{90} = \begin{bmatrix} 1.513*10^6 & 0.514*10^6 & 0 \\ 0.514*10^6 & 20.17*10^6 & 0 \\ 0 & 0 & 1*10^6 \end{bmatrix}$$

$$[\bar{Q}]^{45} = \begin{bmatrix} 6.68*10^6 & 4.68*10^6 & 4.67*10^6 \\ 4.68*10^6 & 6.68*10^6 & 4.67*10^6 \\ 4.67*10^6 & 4.67*10^6 & 5.16*10^6 \end{bmatrix}, [\bar{Q}]^{-45} = \begin{bmatrix} 6.68*10^6 & 4.68*10^6 & -4.67*10^6 \\ 4.68*10^6 & 6.68*10^6 & -4.67*10^6 \\ -4.67*10^6 & -4.67*10^6 & 5.16*10^6 \end{bmatrix}$$

Note how the $\bar{Q}_{16}$ and $\bar{Q}_{26}$ terms are equal and opposite in $[\bar{Q}]^{45}$ and $[\bar{Q}]^{-45}$, this is why a balanced laminate will not have shear-extension coupling; the terms will cancel out if there equal numbers of $\pm\,\theta$

Calculating the A, B, D matrix

After we have the $\overline{\mathrm{Q}}$ matrix for each ply, we can assemble the A, B and D matrices which are defined as

$$[A] = \sum_{k-1}^{n} [\bar{Q}]_k (z_k - z_{k-1}) = \sum_{k-1}^{n} [\bar{Q}]_k t_k \quad, [B] = \frac{1}{2}\sum_{k-1}^{n} [\bar{Q}]_k \left(z_k^2 - z_{k-1}^2\right) \quad, [D] = \frac{1}{3}\sum_{k-1}^{n} [\bar{Q}]_k \left(z_k^3 - z_{k-1}^3\right)$$

In order to calculate these matrices, we need the appropriate values of z_k. To find the z positions we need to know the total thickness of the laminate and the thickness of each lamina. The total thickness is the sum of all the lamina thicknesses. In this case, all of those are the same, so

$$h = 4(lamina) * 2(Symmetric) * 0.0075\left(\frac{in}{lamina}\right) = 0.06\ inches$$

Since z=0 is defined to be along the midplane, and z_0 is negative, we can then define z_0 and z_n

$$z_0 = -\frac{h}{2} = -\frac{0.03}{2} = -0.030\ inches, \qquad z_n = \frac{h}{2} = 0.030\ inches$$

Then, the rest of the z values can be filled in. The desired z values are the top and bottom of each laminate, so the next value in sequence is simply the addition of the previous z value and the thickness of the lamina in question;

$$z_{k+1} = z_k + t_k$$

Thus we have

$$z_0 = -0.030, z_1 = -0.030 + 0.0075 = -0.0225, z_2 = -0.015, z_3 = -.0075, z_4 = 0$$

$$z_5 = 0.0075\ , z_5 = 0.015, z_6 = 0.0225, z_7 = 0.030$$

Having the values of z_k, we can for example find the first entry of the A matrix as;

$$A_{11} = \bar{Q}_{11}^{0} t_0 + \bar{Q}_{11}^{45} t_0 + \bar{Q}_{11}^{90} t_0 + \bar{Q}_{11}^{-45} t_0 + \bar{Q}_{11}^{-45} t_0 + \bar{Q}_{11}^{90} t_0 + \bar{Q}_{11}^{45} t_0 + \bar{Q}_{11}^{0} t_0$$

For convenient simplification of complicated laminates are symmetric, you can calculate the first half of the A matrix entries and then simply multiply by two

$$A_{11} = \left(\bar{Q}_{11}^{0} t_0 + \bar{Q}_{11}^{45} t_0 + \bar{Q}_{11}^{90} t_0 + \bar{Q}_{11}^{-45} t_0\right) * 2$$

This ends up being;

$$A_{11} = \big(20.17 * 10^6 * (0.0075) + 6.68 * 10^6 * (0.0075) \ + 1.513 * 10^6 * (0.0075) + 6.68 * 10^6 * (0.0075)\big) * 2$$

$$A_{11} = 3.13 * 10^5 \frac{lb}{in}$$

Note that the superscript is defining which ply $[\bar{Q}]$ matrix the value is coming from while the subscript defines the appropriate entry. As well, although the superscript is the angle of the ply used to calculate the $[\bar{Q}]$ matrix, not the explicit value of k which in this case would vary from 0 to 4 or 0 to 8 .

Calculating the rest of the values using the same methodology;

$$[ABD] = \begin{bmatrix} 5.26*10^5 & 1.56*10^5 & 0 & 0 & 0 & 0 \\ 1.56*10^5 & 5.26*10^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.85*10^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.50*10^3 & 3.27*10^1 & 2.36*10^1 \\ 0 & 0 & 0 & 3.27*10^1 & 9.30*10^1 & 2.36*10^1 \\ 0 & 0 & 0 & 2.36*10^1 & 2.36*10^1 & 4.14*10^1 \end{bmatrix}$$

Finding the effective thermal and moisture forces and moments

Because we have no change in temperature or moisture content, we can neglect the calculations necessary to find the thermal and moisture forces/moments, all of these calculations will be zero.

Calculation of Midplane Strains and Curvatures

Now we can assemble the entire system, the total applied loading is the sum of the mechanical, thermal and moisture loads;

$$\begin{bmatrix} N \\ M \end{bmatrix}^{Total} = \begin{bmatrix} N \\ M \end{bmatrix}^{Mechanical} + \begin{bmatrix} N \\ M \end{bmatrix}^{Thermal} + \begin{bmatrix} N \\ M \end{bmatrix}^{Moisture}$$

$$\begin{bmatrix} N \\ M \end{bmatrix}^{Total} = \begin{bmatrix} 300 \\ 100 \\ 0 \\ 5.0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 300 \\ 100 \\ 0 \\ 5.0 \\ 0 \\ 0 \end{bmatrix}$$

Using a numerical tool (or basic linear algebra) to invert the 6x6 ABD matrix, we obtain the abd matrix which can be used to determine the midplane strains and curvatures by multiplying;

$$[abd] \begin{bmatrix} N \\ M \end{bmatrix}^{Total} = \begin{bmatrix} \epsilon \\ \kappa \end{bmatrix}$$

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} 5.64*10^{-4} \\ 2.3*10^{-5} \\ 0 \\ 2.15*10^{-2} \\ -5.197*10^{-3} \\ -9.304*10^{-3} \end{bmatrix}$$

Finding the total strain at any point in the thickness of the laminate

These can then be utilized to find the strain at any point through the expression

$$\epsilon = \epsilon^0 + z*\kappa$$

For example, to find the strain along the upper surface, we have $z = -0.01875$ thus the strains are

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}^{Top\ Surface} = \begin{bmatrix} 5.64*10^{-4} \\ 2.3*10^{-5} \\ 0 \end{bmatrix} + (-0.030)*\begin{bmatrix} 2.15*10^{-2} \\ -5.19*10^{-3} \\ -9.304*10^{-3} \end{bmatrix} = \begin{bmatrix} -8.2 \\ 17.9 \\ 27.9 \end{bmatrix}*10^{-5}$$

Finding the "effective properties"

Additionally, $[abd]$ can be employed to find the effective laminate properties which have been defined as;

$$E_x = \frac{1}{h * a_{11}} = 7.99 * 10^6 psi, \qquad E_y = \frac{1}{h * a_{22}} = 7.99 * 10^6 psi\ ,$$

$$G_{xy} = \frac{1}{h * a_{66}} = 3.08 * 10^6\ psi\ , \qquad \nu_{12} = -\frac{a_{21}}{a_{11}} = 0.296$$

Also, as stacking sequence strongly effects the bending stiffness of the laminate, the effective flexural stiffness is also calculated.

$$E_x^f = \frac{12}{h^3 d_{11}} = 12.9 * 10^6 psi, \qquad E_y^f = \frac{12}{h^3 d_{22}} = 4.33 * 10^6 psi,$$

$$G_{xy}^f = \frac{12}{h^3 d_{66}} = 1.59 * 10^5\ psi, \qquad \nu_{12}^f = -\frac{d_{21}}{d_{11}} = 0.242$$

Note that this is a quasi-isotropic laminate, but this only applies to the in plane stiffness values, the flexural stiffness in the two directions is not identical. As discussed previously, the flexural stiffness is dependent on the stacking sequence, which results in different D_{11} and D_{22} values.

Example of CLT 2

Introductory remarks

The below is an example of all the calculations necessary for a general case to calculate the deformation of a laminate undergoing mechanical loading, temperature changes and moisture absorption. Recognizing special cases can reduce the calculations necessary to perform the predictions, e.g. if there is no moisture absorption, the entirety of those calculations can be neglected.

As well, if the laminate is symmetric, the B matrix does not need to be calculated, it will be 0 and if the laminate is balanced, $A_{16} = A_{26} = D_{16} = D_{26} = 0$. Similarly, if the change in temperature or moisture is zero, the fictitious forces and moments can be omitted. Finally, all the matrices are symmetric, thus only the upper triangle is calculated, the lower triangle is identical and therefore is simply reflected across the diagonal.

Problem Statement

If a (0/45/90/45/-45) which is cured at $350°F$ and is now in service at $80°F$ and has absorbed 1% moisture, what will the resultant strains and curvatures, assuming a loading state of;

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} 300\frac{lb}{in} \\ 100\frac{lb}{in} \\ 200\frac{lb}{in} \\ 5\frac{lbin}{in} \\ 3\frac{lbin}{in} \\ 1\frac{lbin}{in} \end{bmatrix}$$

Assume the lamina material properties are;

$$E_1 = 20 * 10^6\ psi\ E_2 = 1.5 * 10^6 psi, G_{12} = 1 * 10^6 psi, \nu_{12} = .34$$

$$\alpha_1 = 0.2 * 10^{-6}\frac{}{°F}, \alpha_2 = 20 * 10^{-6}\frac{}{°F}, \beta_1 = 0.2 * 10^{-4}, \beta_2 = 20 * 10^{-4}, thickness = 0.0075\ in$$

Solution:

Finding the Q and $\bar{Q}$ matrices

To solve this problem, we first need to find the Q matrix which is defined as:

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{62} & Q_{66} \end{bmatrix} = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

In order to evaluate this expression, we first need to find ν_{21}. From before, the expression is

$$\nu_{21} = \nu_{12} * \frac{E_2}{E_1} = 0.34 * \left(\frac{1.5}{20}\right) = 0.0255$$

Inserting the appropriate values:

$$[Q] = \begin{bmatrix} \frac{20*10^6}{1-0.34*0.0255} & \frac{0.34*1.5*10^6}{1-0.34*0.0255} & 0 \\ \frac{0.34*1.5*10^6}{1-0.34*0.0255} & \frac{1.5*10^6}{1-0.34*0.0255} & 0 \\ 0 & 0 & 1*10^6 \end{bmatrix} = \begin{bmatrix} 20.17*10^6 & 0.514*10^6 & 0 \\ 0.514*10^6 & 1.513*10^6 & 0 \\ 0 & 0 & 1*10^6 \end{bmatrix} psi$$

Then, for each unique ply, the $\overline{Q}$ matrix needs to be found. The expressions are:

$$\bar{Q}_{11} = Q_{11}\cos^4\theta \; + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$

$$\bar{Q}_{12} = \bar{Q}_{21} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta)$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta \; + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta$$

$$\bar{Q}_{16} = \bar{Q}_{61} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta$$

$$\bar{Q}_{26} = \bar{Q}_{62} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^2\theta$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta)$$

From trigonometry, for the 0° ply, $[\bar{Q}] = [Q]$, and for the 90° ply, since we have $\cos 90° = 0, \sin 90° = 1$, only the Q_{11} and Q_{22} expressions are reversed. Evaluating the expressions for $\theta = 0°, 90°, 45°, -45°$ the four $[\bar{Q}]$ matrices are thus:

$$[\bar{Q}]^0 = \begin{bmatrix} 20.17*10^6 & 0.514*10^6 & 0 \\ 0.514*10^6 & 1.513*10^6 & 0 \\ 0 & 0 & 1*10^6 \end{bmatrix}, [\bar{Q}]^{90} = \begin{bmatrix} 1.513*10^6 & 0.514*10^6 & 0 \\ 0.514*10^6 & 20.17*10^6 & 0 \\ 0 & 0 & 1*10^6 \end{bmatrix}$$

$$[\bar{Q}]^{45} = \begin{bmatrix} 6.68*10^6 & 4.68*10^6 & 4.67*10^6 \\ 4.68*10^6 & 6.68*10^6 & 4.67*10^6 \\ 4.67*10^6 & 4.67*10^6 & 5.16*10^6 \end{bmatrix}, [\bar{Q}]^{-45} = \begin{bmatrix} 6.68*10^6 & 4.68*10^6 & -4.67*10^6 \\ 4.68*10^6 & 6.68*10^6 & -4.67*10^6 \\ -4.67*10^6 & -4.67*10^6 & 5.16*10^6 \end{bmatrix}$$

Note how the $\bar{Q}_{16}$ and $\bar{Q}_{26}$ terms are equal and opposite in $[\bar{Q}]^{45}$ and $[\bar{Q}]^{-45}$, this is why a balanced laminate will not have shear-extension coupling; the terms will cancel out if there equal numbers of $\pm\,\theta$

Calculating the A, B, D matrix

After we have the $\overline{Q}$ matrix for each ply, we can assemble the A, B and D matrices which are defined as

$$[A] = \sum_{k-1}^{n}[\bar{Q}]_k(z_k - z_{k-1}), \qquad [B] = \frac{1}{2}\sum_{k-1}^{n}[\bar{Q}]_k\left(z_k^2 - z_{k-1}^2\right), \qquad [D] = \frac{1}{3}\sum_{k-1}^{n}[\bar{Q}]_k\left(z_k^3 - z_{k-1}^3\right)$$

In order to calculate these matrices, we need the appropriate values of z_k. To find the z positions we need to know the total thickness of the laminate and the thickness of each lamina. The total thickness is the sum of all the lamina thicknesses. In this case, all of those are the same, so

$$h = 5(lamina) * 0.0075\left(\frac{in}{lamina}\right) = 0.0375\ inches$$

Since z=0 is defined to be along the midplane, and z_0 is negative, we can then define z_0 and z_n

$$z_0 = -\frac{h}{2} = -\frac{0.0375}{2} = -0.01875\ inches, \qquad z_n = \frac{h}{2} = .01875\ inches$$

Then, the rest of the z values can be filled in. The desired z values are the top and bottom of each laminate, so the next value in sequence is simply the addition of the previous z value and the thickness of the lamina in question;

$$z_{k+1} = z_k + t_k$$

Thus we have

$$z_0 = -0.01875, z_1 = -0.01875 + 0.0075 = -0.01125, z_2 = -0.00375$$

$$z_3 = 0.00375, z_4 = 0.01125, z_5 = 0.01875$$

Note that none of the values are $z_k = 0$. This is because there is an odd number of plies, the centerline of the 90 ply lies along the centerline and since we only care about the z positions on the edges of each ply, we ignore this location.

Having the values of z_k, we can for example find the first entry of the A matrix as;

$$A_{11} = \bar{Q}_{11}^{0}(z_1 - z_0) + \bar{Q}_{11}^{45}(z_2 - z_1) + \bar{Q}_{11}^{90}(z_3 - z_2) + \bar{Q}_{11}^{45}(z_4 - z_3) + \bar{Q}_{11}^{-45}(z_5 - z_4)$$

$$A_{11} = 20.17 * 10^6 * \big(-0.01125 - (-0.01875)\big) + 6.68 * 10^6 * \big(-0.00375 - (-0.01125)\big) + 1.513 * 10^6 * \big(0.00375 - (-0.00375)\big) + 6.68 * 10^6 * (0.01125 - 0.00375) + 6.68 * 10^6 * (0.01875 - 0.01125)$$

$$A_{11} = 20.17 * 10^6 * 0.0075 + 6.68 * 10^6 * 0.0075 + 1.513 * 10^6 * 0.0075 + 6.68 * 10^6 * 0.0075 + 6.68 * 10^6 * 0.0075$$

Note that the superscript is defining which $[\bar{Q}]$ matrix the value is coming from while the subscript defines the appropriate entry. As well, although the superscript is the angle of the ply used to calculate the $[\bar{Q}]$ matrix, not the explicit value of k which in this case would vary from 0 to 5.

As you may have noted, $z_k - z_{k-1}$ is equal to the thickness of each lamina. For consistency with the B and D matrix expressions the difference expression is retained, although for efficiency, simply multiplying the appropriate $[\bar{Q}]$ term by the thickness of the ply in question will suffice. This shortcut cannot be used for $[B]$and $[D]$. Calculating the value;

$$A_{11} = 3.13 * 10^5\,\frac{lb}{in}$$

Calculating the rest of the values using the same methodology;

$$[ABD] = \begin{bmatrix} 3.13*10^5 & 1.13*10^5 & 3.50*10^4 & -1.52*10^3 & 4.69*10^2 & -5.25*10^2 \\ 1.13*10^5 & 3.13*10^5 & 3.50*10^5 & 4.69*10^2 & 5.81*10^2 & -5.25*10^2 \\ 3.50*10^4 & 3.50*10^4 & 1.31*10^5 & -5.25*10^2 & -5.25*10^2 & 4.69*10^2 \\ -1.52*10^2 & 4.69*10^2 & -5.25*10^2 & 52.4 & 13.2 & -3.77 \\ 4.69*10^2 & 5.81*10^2 & -5.25*10^2 & 13.2 & 20.93 & -3.77 \\ -5.25*10^2 & -5.25*10^2 & 4.69*10^2 & -3.77 & -.377 & 15.4 \end{bmatrix}$$

Finding the effective thermal and moisture forces and moment

Now to find the thermal "forces/moments," to apply to the laminate we use the expressions which have been previously defined as;

$$\begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix} = \sum_{k-1}^{n} [\bar{Q}]_k \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k \Delta T(z_k - z_{k-1}), \qquad \begin{bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix} = \frac{1}{2}\sum_{k-1}^{n} [\bar{Q}]_k \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k \Delta T\left(z_k^2 - z_{k-1}^2\right)$$

For example, to find the thermal forces, we have;

$$\begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix} = \Delta T * \left[[\bar{Q}]_0 \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_0 (z_1 - z_0) + [\bar{Q}]_{45} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_{45} (z_2 - z_1) + [\bar{Q}]_{90} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_{90} (z_3 - z_2) \right.$$
$$\left. + [\bar{Q}]_{45} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_{45} (z_4 - z_3) + [\bar{Q}]_{-45} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_{-45} (z_5 - z_4) \right]$$

Note that the x-y transformed coefficients of thermal expansion are necessary for these calculations, which have been previously defined as

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix} = \begin{bmatrix} \alpha_1 \cos^2\theta + \alpha_2 \sin^2\theta \\ \alpha_1 \sin^2\theta + \alpha_2 \cos^2\theta \\ 2*\cos\theta\sin\theta\,(\alpha_1 - \alpha_2) \end{bmatrix}$$

Utilizing $\theta = 0, 45, -45, 90$, we have

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_0 = \begin{bmatrix} 0.2*10^{-6} \\ 20*10^{-6} \\ 0 \end{bmatrix} \frac{}{{}^\circ F}, \qquad \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_{45} = \begin{bmatrix} 1.01*10^{-5} \\ 1.01*10^{-5} \\ -1.98*10^{-5} \end{bmatrix} \frac{}{{}^\circ F},$$

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_{-45} = \begin{bmatrix} 1.01*10^{-5} \\ 1.01*10^{-5} \\ -1.98*10^{-5} \end{bmatrix} \frac{}{{}^\circ F}, \qquad \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_{90} = \begin{bmatrix} 20*10^{-6} \\ 0.2*10^{-6} \\ 0 \end{bmatrix} \frac{}{{}^\circ F},$$

Then, carrying out the matrix multiplication to find the thermal forces and moments we have

$$\begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \\ M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix} = \begin{bmatrix} -226.2 \\ -226.2 \\ 16.2 \\ -0.24 \\ 0.24 \\ -0.24 \end{bmatrix}$$

Following the same procedure to find the moisture "forces/moments" defined as

$$\begin{bmatrix} N_x^m \\ N_y^m \\ N_{xy}^m \end{bmatrix} = \sum_{k-1}^{n} [\bar{Q}]_k \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix}_k \Delta\mathrm{m}(z_k - z_{k-1}), \qquad \begin{bmatrix} M_x^m \\ M_y^m \\ M_{xy}^m \end{bmatrix} = \frac{1}{2} \sum_{k-1}^{n} [\bar{Q}]_k \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix}_k \Delta\mathrm{m}(z_k^2 - z_{k-1}^2)$$

We obtain

$$\begin{bmatrix} N_x^m \\ N_y^m \\ N_{xy}^m \\ M_x^m \\ M_y^m \\ M_{xy}^m \end{bmatrix} = \begin{bmatrix} 83.8 \\ 83.8 \\ -6.0 \\ 0.09 \\ -0.09 \\ 0.09 \end{bmatrix}$$

Calculation of Midplane Strains and Curvatures

Now we can assemble the entire system, the total applied loading is the sum of the mechanical, thermal and moisture loads;

$$\begin{bmatrix} N \\ M \end{bmatrix}^{Total} = \begin{bmatrix} N \\ M \end{bmatrix}^{Mechanical} + \begin{bmatrix} N \\ M \end{bmatrix}^{Thermal} + \begin{bmatrix} N \\ M \end{bmatrix}^{Moisture}$$

$$\begin{bmatrix} N \\ M \end{bmatrix}^{Total} = \begin{bmatrix} 300 \\ 100 \\ 200 \\ 5.0 \\ 3.0 \\ 1.0 \end{bmatrix} + \begin{bmatrix} -226.2 \\ -226.2 \\ 16.2 \\ -0.24 \\ 0.24 \\ -0.24 \end{bmatrix} + \begin{bmatrix} 83.8 \\ 83.8 \\ -6.0 \\ 0.09 \\ -0.09 \\ 0.09 \end{bmatrix} = \begin{bmatrix} 157.6 \\ -42.4 \\ 210.2 \\ 4.85 \\ 3.15 \\ 0.85 \end{bmatrix}$$

Using a numerical tool (or basic linear algebra) to invert the 6x6 ABD matrix, we obtain the abd matrix which can be used to determine the midplane strains and curvatures by multiplying; $[abd]\begin{bmatrix} N \\ M \end{bmatrix}^{Total} = \begin{bmatrix} \epsilon \\ \kappa \end{bmatrix}$

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} 1.10 * 10^{-3} \\ -1.23 * 10^{-3} \\ 2.61 * 10^{-3} \\ 1.26 * 10^{-1} \\ 1.53 * 10^{-1} \\ 3.95 * 10^{-2} \end{bmatrix}$$

Finding the total strain at any point in the thickness of the laminate

These can then be utilized to find the strain at any point through the expression

$$\epsilon = \epsilon^0 + z * \kappa$$

For example, to find the strain along the upper surface, we have $z = -0.01875$ thus the strains are

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}^{Top\ Surface} = \begin{bmatrix} 1.10 * 10^{-3} \\ -1.23 * 10^{-3} \\ 2.61 * 10^{-3} \end{bmatrix} + (-0.01875) * \begin{bmatrix} 1.26 * 10^{-1} \\ 1.53 * 10^{-1} \\ 3.95 * 10^{-2} \end{bmatrix} = \begin{bmatrix} 1.25 * 10^{3} \\ -4.11 * 10^{3} \\ 1.87 * 10^{-3} \end{bmatrix}$$

Finding the "effective properties"

Additionally, $[abd]$ can be employed to find the effective laminate properties which have been defined as;

$$E_x = \frac{1}{h * a_{11}} = 4.57 * 10^6 psi, \qquad E_y = \frac{1}{h * a_{22}} = 6.15 * 10^6 psi\ ,$$

$$G_{xy} = \frac{1}{h * a_{66}} = 2.39 * 10^6\ psi\ , \qquad \nu_{12} = -\frac{a_{21}}{a_{11}} = 0.244$$

Also, as stacking sequence strongly effects the bending stiffness of the laminate, the effective flexural stiffness is also calculated.

$$E_x^f = \frac{12}{h^3 d_{11}} = 6.91 * 10^6 psi, \qquad E_y^f = \frac{12}{h^3 d_{22}} = 2.95 * 10^6 psi,$$

$$G_{xy}^f = \frac{12}{h^3 d_{66}} = 2.57 * 10^6\ psi, \qquad \nu_{12}^f = -\frac{d_{21}}{d_{11}} = 0.716$$

Of particular note is that the effective properties are derived assuming a symmetric, balanced laminate, which this is not. This assumption is necessary because the effective properties assume that there is no coupling between extension, shear and bending. Thus, these effective properties can be used to give an impression of how stiff the laminate is but does not account for the coupling effects.

